UNIT - II

Cauchy's Problem for first order Equation.

Complete Solution (or) Complete integral:-

The relation of the type $F(x, y, z, a, b) = 0$ leads to partial differential eqn's of the first order. Any such relation which contains two arbitrary constants 'a' and 'b' is soln of P.D.E of first order is said to be a complete solution or complete integral of that eqn.

General Solution (or) General integral:-

Any relation of the type $f(u,v) = 0$ involving an arbitrary function 'f' connecting to two known functions 'u' and 'v' of the function 'f' and providing a soln of first order partial differential eqn is called a general solution or general integral of first order.

Sec- Linear Equations of First Order

Lagrange's Eqn:- A linear partial differential equation of first order is of the form $Pp + Qq = R$ where P, Q, R are functions of x, y, z such that a partial differential equation is known as Lagrange's Equation.

Theorem:- ① The general solution of the linear partial differential eqn $Pp + Qq = R$ is $\phi(u,v) = 0$ where 'ϕ' is an arbitrary function and $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ forms a solution of the equations $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Proof:- Given equation is $Pp + Qq = R$ —(1)

claim: the general soln of ① is $\phi(u,v) = 0$ where $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ forms a soln of the equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

In this theorem, we prove that if $\phi(u,v) = 0$ is a soln then we obtain the P.D.E $Pp + Qq = R$

where $u(x,y,z) = c_1$ and

$v(x,y,z) = c_2$

Let us consider the sol^n of eq^n (1) is $\phi(u,v)=0$ —(2)

we have $\left.\begin{array}{l} u(x,y,z)=c_1 \\ v(x,y,z)=c_2 \end{array}\right\}$ -(3) are the sol^{n}s of the eq^n.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad —(4)$$

Now differentiate eq^n (2) with respect to 'x' partially we get

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+\frac{\partial z}{\partial x}\cdot\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+\frac{\partial z}{\partial x}\frac{\partial v}{\partial z}\right]=0$$

we know that $\frac{\partial z}{\partial x}=p$

$$\Rightarrow \frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right]=0 \quad —(5)$$

Now differentiate eq^n (2) with respect to 'y' partially we get

$$\frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+\frac{\partial z}{\partial y}\cdot\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}+\frac{\partial z}{\partial y}\frac{\partial v}{\partial z}\right]=0$$

we know that $\frac{\partial z}{\partial y}=q$

$$\Rightarrow \frac{\partial\phi}{\partial u}\left[\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right]+\frac{\partial\phi}{\partial v}\left[\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right]=0 \quad —(6)$$

Now Eliminate $\frac{\partial\phi}{\partial u}$ and $\frac{\partial\phi}{\partial v}$ from eq^n's (5) & (6) we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z} & \frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z} & \frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z} \end{vmatrix}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right)-\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)=0$$

$$\Rightarrow \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+q\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}+p\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}+\cancel{pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}-q\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-p\frac{\partial v}{\partial z}\frac{\partial u}{\partial y}-\cancel{pq\frac{\partial v}{\partial z}\frac{\partial u}{\partial z}}$$

$$\Rightarrow p\left[\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial z}\frac{\partial u}{\partial y}\right]+q\left[\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}\right]+\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right)=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left[\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial v}{\partial z}\frac{\partial u}{\partial x}\right]q=\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial v}{\partial x}\frac{\partial u}{\partial y}\right) —(7)$$

ie, it is in the form $Pp+Qq=R$

Hence eq^n (2) is a sol^n of eq^n (7)

Now taking the differentiation of 'u' & 'v' of $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ we get

$$\left(\frac{\partial u}{\partial x}\right)dx+\left(\frac{\partial u}{\partial y}\right)dy+\left(\frac{\partial u}{\partial z}\right)dz=0 \quad —(8)$$

$$\left(\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial v}{\partial y}\right)dy+\left(\frac{\partial v}{\partial z}\right)dz=0 \quad —(9)$$

$\because$ 'u' and 'v' are independent functions of 'x' and 'y'

Solving (8) and (9) we have

$$\frac{dx}{\frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dy}}=\frac{dy}{\frac{dv}{dx}\frac{du}{dz}-\frac{dv}{dz}\frac{du}{dx}}=\frac{dz}{\frac{du}{dx}\frac{dv}{dy}-\frac{dv}{dx}\frac{du}{dy}} \quad -(10)$$

From eq^n's (4) and (10) we obtain

$$\frac{\frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dy}}{P}=\frac{\frac{du}{dz}\frac{dv}{dx}-\frac{dv}{dz}\frac{du}{dx}}{Q}=\frac{\frac{du}{dx}\frac{dv}{dy}-\frac{dv}{dx}\frac{du}{dy}}{R}=K(\text{say})$$

$\Rightarrow \frac{du}{dy}\frac{dv}{dz}-\frac{du}{dz}\frac{dv}{dy}=PK$

$\Rightarrow \frac{du}{dz}\frac{dv}{dx}-\frac{dv}{dz}\frac{du}{dx}=QK$

$\Rightarrow \frac{du}{dx}\frac{dv}{dy}-\frac{dv}{dx}\frac{du}{dy}=RK$

substitude these values in (7) we get

$KPp+KQq=KR$

$\cancel{K}(Pp+Qq)=\cancel{K}R$

$\therefore Pp+Qq=R$

$\therefore u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are two sol^n's of the system of differential eq^n's.

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$\therefore \phi(u,v)=0$ is a solution of $Pp+Qq=R$

where ϕ is an arbitrary function.

Hence proved.

Note:- The eq^n $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ is called Lagrange's auxilary eq^n of $Pp+Qq=R$

Problems:-

1. Find the general sol^n of the diff-al eq^n $x^2\frac{dz}{dx}+y^2\frac{dz}{dy}=(x+y)z$

Given equation:- $x^2\frac{dz}{dx}+y^2\frac{dz}{dy}=(x+y)z$ —(1)

we know that $\frac{dz}{dx}=p$, $\frac{dz}{dy}=q$

Substitude these values in (1)

$x^2p+y^2q=(x+y)z$

This eq^n is of the form $Pp+Qq=R$ —(2)

Where $P=x^2$; $Q=y^2$; $R=z(x+y)$

we know that Lagrange's eq^n of (2) is

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{-y(3x^2+y^2)}$$

$$\frac{dy}{dx} = \frac{-y(3x^2+y^2)}{x(x^2+3y^2)}$$

$$= \frac{-y\cancel{x^2}\left(3+\frac{y^2}{x^2}\right)}{\cancel{x^2}\left(1+3\frac{y^2}{x^2}\right)}$$

$$= \frac{-y\left(3+\frac{y^2}{x^2}\right)}{x\left(1+\frac{3y^2}{x^2}\right)}$$

Take $v = \frac{y}{x}$

$$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{-y(3+v^2)}{x(1+3v^2)}$$

$$v + x\frac{dv}{dx} = \frac{-v\cancel{x}(3+v^2)}{\cancel{x}(1+3v^2)}$$

$$x\frac{dv}{dx} = \frac{-3v-v^3}{1+3v^2} - v$$

$$x\frac{dv}{dx} = \frac{-3v-v^3-v-3v^3}{1+3v^2}$$

$$x\frac{dv}{dx} = \frac{-\cancel{3}v^3 - \cancel{v^2} - \cancel{v} \cancel{3}}{1+\cancel{3v^2}} \quad \frac{-4v-4v^3}{1+3v^2}$$

$$x\frac{dv}{dx} = \frac{-4v(1+v^2)}{1+3v^2}$$

$$\frac{1+3v^2}{v(1+v^2)}dv = -4\frac{dx}{x}$$

$$\frac{1+3v^2}{v(1+v^2)}dv + 4\frac{dx}{x} = 0$$

$$\Rightarrow \frac{d(v+v^3)}{(v+v^3)}dv + 4\frac{dx}{x} = 0$$

Integrate on both sides

$$\log|v+v^3| + 4\log x = \log c_2$$

$$\log(v+v^3)(x^4) = \log c_2$$

$$(v+v^3)(x^4) = c_2$$

$$\left(\frac{y}{x}+\frac{y^3}{x^3}\right)x^4 = c_2$$

$$\frac{x^2y+y^3}{\cancel{x^3}}(\cancel{x^4}) = c_2$$

$$xy(x^2+y^2) = c_2$$

w.k.T $v(x,y,z) = c_2$

$v(x,y,z) = xy(x^2+y^2)$

$\therefore$ the general soln of given eqn is $\phi(u,v) = 0$

$\Rightarrow \phi(\frac{xy}{z}, xy(x^2+y^2))$ is the general soln of given diff.- equation

4, solve $z(xp-yq) = y^2-x^2$

Given eqn is $z(xp-yq) = y^2-x^2$ —(1)

(1) is of the form $Pp+Qq=R$

where $P=zx$; $Q=-zy$; $R=y^2-x^2$

we know that Lagrange's eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{zx} = \frac{dy}{-zy} = \frac{dz}{y^2-x^2} \quad \text{—(A)}$$

Taking first two integrals.

$\frac{dx}{\cancel{z}x} = \frac{dy}{-\cancel{z}y}$ Integrate on both sides

$\log x = -\log y + \log c_1$

$xy = c_1$

Taking x, y, z as multipliers of each of above functions, we get

$$\frac{x\,dx + y\,dy + z\,dz}{zx^2 - zy^2 + zy^2 - zx^2} = 0$$

$x\,dx + y\,dy + z\,dz = 0$

Integrate on both sides

$x^2+y^2+z^2 = c_2$

we know that $u(x,y,z) = c_1 \Rightarrow u(x,y,z) = xy$

$v(x,y,z) = c_2 \Rightarrow v(x,y,z) = x^2+y^2+z^2$

$\therefore$ The general soln of eqn (1) is $\phi(u,v) = 0$

$\Rightarrow \phi(xy, x^2+y^2+z^2)$ is the general soln of given eqn.

5, Solve $y^2p - xyq = x(z-2y)$

Given eqn is $y^2p - xyq = x(z-2y)$ —(1)

compare (1) with $Pp+Qq=R$

where $P=y^2$; $Q=-xy$; $R=xz-2xy$

we know that Lagrange's eqn of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz-2xy}$$

Taking first two integrals

$$\frac{dx}{y^2} = \frac{dy}{-xy}$$

$$-x\,dx = y\,dy$$

Integrate on both sides

$$-x^2 = y^2 + c_1$$

$$x^2 + y^2 = c_1$$

we know that $u(x,y,z) = c_1 \Rightarrow u(x,y,z) = x^2+y^2$

Taking second and third integrals we get

$$\frac{dy}{-\cancel{x}y} = \frac{dz}{\cancel{x}(z-2y)}$$

$$\frac{dy}{y} + \frac{d(z-2y)}{z-2y} = 0$$

Integrate on both sides

$$\log y + \log|z-2y| = \log c_2$$

$$y(z-2y) = c_2$$

we know that $v(x,y,z) = c_2$

$$v(x,y,z) = y(z-2y)$$

$\therefore$ The general soln of given eqn $\phi(u,v) = 0$

$$\Rightarrow \phi(x^2+y^2,\ y(z-2y)) = 0$$

3 **Theorem:-** If $u_i(x_1, x_2, x_3 - - - x_n, z) = c_i\ (i = 1,2,3 - - n)$ are independent soln's of the eqn's $\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = - - - \frac{dx_n}{P_n} = \frac{dz}{R}$

Then relation $\Phi(u_1, u_2, u_3 - - u_n) = 0$ in which the function Φ is arbitrary is a general soln of the linear partial differential equation.

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + - - + P_n\frac{\partial z}{\partial x_n} = R$$

Proof:- To prove this theorem, we first of all note that if the solutions of the equations

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} - - - = \frac{dx_n}{P_n} = \frac{dz}{R} \quad \text{—(1) are } u_i(x_1, x_2 - - - x_n, z) = c_i,$$

$i = 1, 2, 3 - - - n$ —(2)

Then the eqn $\sum_{j=1}^{n} \frac{\partial u_i}{\partial x_j}dx_j + \frac{\partial u_i}{\partial z}dz = 0$, $i = 1,2,3 - - n$ —(3) must be compatable with (1)

In other words, we must have $\sum_{j=1}^{n} P_j \frac{\partial u_i}{\partial x_j} + R\frac{\partial u_i}{\partial z} = 0$ —(4)

Solving the set of 'n' equations 4 for P_i, we find that

$$\frac{P_i}{\frac{d(u_1,u_2,u_3\cdots u_n)}{d(x_1,x_2\cdots x_{i-1},z,x_{i+1},\cdots x_n)}} = \frac{R}{\frac{d(u_1,u_2,u_3\cdots u_n)}{d(x_1,x_2,x_3\cdots x_n)}} \quad \text{for } i=1,2,\cdots n \quad —(5)$$

where $\frac{d(u_1,u_2,u_3\cdots u_n)}{d(x_1,x_2,x_3\cdots x_n)}$ denote the Jacobian equation

$$\begin{vmatrix} \frac{du_1}{dx_1} & \frac{du_1}{dx_2} & \cdots & \frac{du_1}{dx_n} \\ \frac{du_2}{dx_1} & \frac{du_2}{dx_2} & \cdots & \frac{du_2}{dx_n} \\ \vdots & \vdots & & \\ \frac{du_n}{dx_1} & \frac{du_n}{dx_2} & \cdots & \frac{du_n}{dx_n} \end{vmatrix}$$

Now consider the relation $\Phi(u_1,u_2,u_3\cdots u_n)=0$ —(6)

Now differentiate ⑥ with respect to x_i we obtain the equation

$$\sum_{j=1}^{n}\left(\frac{d\Phi}{du_j}\frac{du_j}{dx_i}+\frac{du_i}{dz}\frac{dz}{dx_i}\right)=0 \Rightarrow \left(\frac{\partial\Phi}{\partial u_1}\cdot\frac{\partial u_1}{\partial x_i}+\frac{\partial u_1}{\partial z}\frac{\partial z}{\partial x_i}\right)+\left(\frac{\partial\Phi}{\partial u_2}\cdot\frac{\partial u_2}{\partial x_i}+\frac{\partial u_2}{\partial z}\frac{\partial z}{\partial x_i}\right)+\cdots+\left(\frac{\partial\Phi}{\partial u_n}\frac{\partial u_n}{\partial x_i}+\frac{\partial u_n}{\partial z}\frac{\partial z}{\partial x_i}\right)$$

and these are 'n' such ~~eq~~ uations, one for each value of i, eliminating the 'n' equations.

$$\frac{d\Phi}{du_1},\frac{d\Phi}{du_2}\cdots\frac{d\Phi}{du_n}$$

From, these equations we obtain the relation $\frac{d(u_1,u_2,u_3\cdots u_n)}{d(x_1,x_2,x_3\cdots x_n)}+$

$$\sum_{j=1}^{n}\frac{dz}{dx_j}\frac{d(u_1,u_2,u_3\cdots u_{j-1},u_j,u_{j+1}\cdots u_n)}{d(x_1,x_2,x_3\cdots x_{j-1},z,x_{j+1}\cdots x_n)}=0 \quad —(7)$$

Substituting (5) into eq^n(7) we see that the function 'z' defined by the relation ⑥ is a solution of the equation.

$$P_1\frac{dz}{dx_1}+P_2\frac{dz}{dx_2}+\cdots+P_n\frac{dz}{dx_n}=R \quad —(8)$$ as we desired to show.

Hence proved.

SEC → Integral Surfaces Passing Through A Given Curve ←

Let $Pp+Qq=R$ —(1) be the given equation.

we know that the auxilary eqⁿ of (1) is

$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ gives the following two independent solutions.

$$\left.\begin{aligned} u(x,y,z)&=c_1 \\ v(x,y,z)&=c_2 \end{aligned}\right\} \text{—(2)}$$

Suppose we obtain the integral surface which passes through the given curve whose equation in parametric form is given by

$x=x(t),\ y=y(t),\ z=z(t)$

Substitude these values in (2) we get

$$(2) \Rightarrow \left.\begin{aligned} u(x(t),y(t),z(t))&=c_1 \\ v(x(t),y(t),z(t))&=c_2 \end{aligned}\right\} \text{where 't' is a parameter}$$

Now eliminate the parameter 't' in above eqⁿ we get a relation in terms of c_1 and c_2.

Finally we replace c_1 and c_2 by using eqⁿ(2), we obtain the required integral surface.

1. Find the integral surface of the linear partial differential eqⁿ $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$ which contains the straight line $x+y=0$, $z=1$

Given eqⁿ is $x(y^2+z)p-y(x^2+z)q=(x^2-y^2)z$ —(1)

Eqⁿ (1) is of the form $Pp+Qq=R$

where $P=x(y^2+z)$; $Q=-y(x^2+z)$; $R=(x^2-y^2)z$

we know that auxilary equation of (1) is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad \text{—(2)}$$

Given straight line is $x+y=0,\ z=1$ —(3)

Substitude P, Q, R values in eqⁿ (3)

$$\frac{dx}{x(y^2+z)}=\frac{dy}{-y(x^2+z)}=\frac{dz}{z(x^2-y^2)}$$

choose $1/x, 1/y, 1/z$ as multipliers of each of fractions of eqⁿ

$$\Rightarrow \frac{\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz}{\cancel{y^2}+\cancel{z}-\cancel{x^2}-\cancel{z}+\cancel{x^2}-\cancel{y^2}}$$

$$\Rightarrow \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0$$

Integrate on both sides

$\Rightarrow \log x + \log y + \log z = \log c_1$

$\Rightarrow \log xyz = \log c_1$

$\Rightarrow xyz = c_1$ —(4)

choose $x, y, -1$ as multipliers of each of fractions of eq^n(A)

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{\cancel{x^2y^2} + \cancel{x^2z} - \cancel{y^2x^2} - \cancel{y^2z} - \cancel{zx^2} + \cancel{zy^2}}$$

$\Rightarrow x\,dx + y\,dy - dz = 0$

Integrate on both sides

$\Rightarrow \frac{x^2}{2} + \frac{y^2}{2} - z = \frac{c_2}{2}$

$\Rightarrow x^2 + y^2 - 2z = c_2$ —(5)

Taking 't' as the parameter then the straight line can be put in the parameter form

$\Rightarrow x + y = 0$, $z = 1$

$\Rightarrow x = -y$

Let $x = t$ then $y = -t$

$z = 1$

From eq^n (4), we have $xyz = c_1$

$\Rightarrow t(-t)(1) = c_1$

$\Rightarrow -t^2 = c_1$

$\Rightarrow t^2 = -c_1$ —(6)

From eq^n(5) we have $x^2 + y^2 - 2z = c_2$

$\Rightarrow (t)^2 + (-t)^2 - 2(1) = c_2$

$\Rightarrow 2t^2 - 2 = c_2$

Substitude $t^2 = -c_1$ (from 6)

$\Rightarrow 2(-c_1) - 2 = c_2$

$\Rightarrow -2c_1 - 2 = c_2$ —(7)

Substitude c_1, c_2 values in above eq^n(7)

$\Rightarrow -2(xyz) - 2 = x^2 + y^2 - 2z$

$\Rightarrow x^2 + y^2 + 2xy - 2z + 2 = 0$ is the required integral surf-ace of given partial differential equation.

3. Find the equation of the integral surface of the differential eq^n $2y(z-3)p+(2x-z)q = y(2x-3)$ which passes through the circle $z=0$, $x^2+y^2=2x$

Given eq^n is $2y(z-3)p+(2x-z)q = y(2x-3)$ — (1) and given circle is $x^2+y^2=2x$, $z=0$ — (2)

compare (1) with $Pp+Qq=R$

where $P=2y(z-3)$; $Q=2x-z$; $R=y(2x-3)$

we know that auxilary eq^n of (1) is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

Substitude P, Q, R values in above eq^n

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \quad \text{—(A)}$$

consider first and third fractions of eq^n(A)

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$\frac{dx}{2(z-3)} = \frac{dz}{(2x-3)}$$

$$(2x-3)dx = (2z-6)dz$$

$$2x\,dx - 3dx = 2z\,dz - 6dz$$

Integrate on both sides

$$\cancel{2}\left(\frac{x^2}{\cancel{2}}\right) - 3x = \cancel{2}\left(\frac{z^2}{\cancel{2}}\right) - 6z + c_1$$

$$x^2 - z^2 - 3x + 6z = c_1 \quad \text{— (3)}$$

Now choose 1, $2y$, -2 as multipliers of eq^n(A)

$$\frac{dx + 2y\,dy - 2dz}{2y\cancel{z} - 6\cancel{y} + 4x\cancel{y} - 2x\cancel{y}z - 4\cancel{x}y + 6\cancel{y}}$$

$$dx + 2y\,dy - 2dz = 0$$

Integrate on both sides

$$x + y^2 - 2z = c_2 \quad \text{— (4)}$$

Given circle is $z=0$, $x^2+y^2=2x$

Substitude $x=t$

$\Rightarrow t^2 + y^2 = 2t$

$\Rightarrow y^2 = 2t - t^2$

substitude y^2 value in eq^n(4)

$$x + 2t - t^2 - 2z = c_2$$

substitude $x=t$ and $z=0$ in above eqⁿ

$$t+2t-t^2=c_2 \quad -(5)$$

substitude $x=t$ & $z=0$ in eqⁿ (3)

$$t^2-3t=c_1 \quad -(6)$$

From eqⁿ (5) we have $c_2 = 3t-t^2$

$$= -(t^2-3t)$$

$$c_2 = -c_1$$

$$\Rightarrow c_1+c_2=0 \quad (\because \text{by } (6))$$

$\therefore$ From (3) & (4) we have $x^2-z^2-3x+6z=c_1$, $x+y^2-2z=c_2$

$\Rightarrow x^2-z^2-3x+6z+x+y^2-2z=0$

$\Rightarrow x^2+y^2-z^2-2x+4z=0$ is the required integral surface of differential equation

3. Find the general integral of the P.D.eqⁿ $(2xy-1)p+(z-2x^2)q=2(x-yz)$ and also the particular integral which passes through the line $x=1$, $y=0$

sol: Given eqⁿ is :- $(2xy-1)p+(z-2x^2)q=2(x-yz)$ —(1)

compare (1) with $Pp+Qq=R$

where $P=2xy-1$; $Q=z-2x^2$; $R=2x-2yz$

we know that auxilary eqⁿ of (1) is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \quad -(2)$$

Substitude P, Q, R values in above eqⁿ

$$\frac{dx}{2xy-1}=\frac{dy}{z-2x^2}=\frac{dz}{2x-2yz} \quad -(A)$$

Now choose $z, 1, x$ as multipliers of eqⁿ (A)

$$\frac{z\,dx+dy+x\,dz}{2x\cancel{y}z-\cancel{z}+\cancel{z}-2\cancel{x^2}+2\cancel{x^2}-2x\cancel{y}z}$$

$$z\,dx+dy+x\,dz=0$$

Integrate on both sides

$$xz+y=c_1 \quad -(3)$$

choose $x, y, 1/2$ as multipliers of eqⁿ (A)

$$\frac{x\,dx+y\,dy+1/2\,dz}{2x^2y-x+zy-2x^2y+x-yz}$$

$$x\,dx+y\,dy+1/2\,dz=0$$

Integrate on both sides $x^2+y^2+z=c_2$ —(4)

Given $x=1, y=0$

Since the values of x, y are given. so consider 't' as the para-
-ter such that $z=t$

$\therefore z=t, x=1, y=0$

From (3) we have $xz+y=c_1$

$\Rightarrow 1(t)+0=c_1$

$\Rightarrow t=c_1$

From (4) we have $x^2+y^2+z=c_2$

$\Rightarrow 1+0+t=c_2$

$\Rightarrow 1+t=c_2$

Substitude $t=c_1$

$\Rightarrow 1+c_1=c_2$

Substitude c_1, c_2 in above eq^n

$1+xz+y=x^2+y^2+z$

$\therefore x^2+y^2-xz-y+z-1=0$

4. Find the integral surface of the eq^n $(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z$ through the curve $xz=a^3, y=0$

Given eq^n is $(x-y)y^2p+(y-x)x^2q=(x^2+y^2)z$ —(1)

and curve is $xz=a^3, y=0$ —(2)

compare (1) with $Pp+Qq=R$, where —(3)

$P=(x-y)y^2$; $Q=(y-x)x^2$; $R=(x^2+y^2)z$

we know that auxilary eq^n of (1) is

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{(x-y)y^2}=\frac{dy}{(y-x)x^2}=\frac{dz}{(x^2+y^2)z} \quad \text{—(A)}$$

Taking first and second fractions.

$$\frac{dx}{(x-y)y^2}=\frac{dy}{(y-x)x^2}$$

$$\frac{dx}{\cancel{(x-y)}y^2}=\frac{dy}{-\cancel{(x-y)}x^2}$$

$$x^2\,dx=-y^2\,dy$$

Integrate on both sides

$$\frac{x^3}{3}+\frac{y^3}{3}=\frac{c_1}{3}$$

$$x^3+y^3=c_1 \quad \text{—(4)}$$

consider, from eq^n (A) $\dfrac{dx-dy}{(x-y)y^2-(y-x)x^2}=\dfrac{dz}{(x^2+y^2)z}$

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{(x^2+y^2)z}$$

$$\frac{d(x-y)}{(x-y)} = \frac{dz}{z}$$

Integrate on both sides

$$\log\left|\frac{x-y}{1}\right| = \log z + \log c_2$$

$$\frac{x-y}{z} = c_2 \quad -(5)$$

Taking 't' as parameter, the given curve can be put in the parametric form as:-

given $y=0$ and substitude $x=t$

since $xz = a^3$

$$tz = a^3$$

$$\boxed{z = \frac{a^3}{t}}$$

From eq^n (4) we have, $x^3+y^3 = c_1$

$$t^3 + 0 = c_1$$

$$\boxed{c_1 = t^3}$$

From eq^n (5), $\frac{x-y}{z} = c_2$

$$\frac{t-0}{\frac{a^3}{t}} = c_2 \quad \Rightarrow \frac{t^2}{a^3} = c_2$$

Multiply with 't' on bothsides

$$\frac{t^3}{a^3} = t\,c_2$$

$$\frac{c_1}{a^3} = t\,c_2$$

$$\Rightarrow c_1 = a^3 t\, c_2$$

Substitude c_1, c_2 values in above eq^n

$$x^3+y^3 = a^3\left(\frac{x-y}{z}\right)t$$

$$\Rightarrow x^3+y^3 = \frac{a^3(x-y)x}{z} \quad (\because t = x)$$

$$\Rightarrow (x^3+y^3)z = a^3(x^2-yx)$$

$\therefore x^3z + y^3z - a^3x^2 + a^3yx = 0$ is the required surface integral.

x (wrong)

6) Find the general integral of the eq^n $(x-y)p + (y-x-z)q = z$ and the particular sol^n to the circle $z=1$, $x^2+y^2=1$

Given eq^n is $(x-y)p + (y-x-z)q = z$ —(1)

and the circle is $z=1$, $x^2+y^2=1$ —(2)

compare (1) with $Pp + Qq = R$ —(3) where

$P = x-y$; $Q = y-x-z$; $R = z$

→ Surfaces Orthogonal To a Given System of Curves

Let $f(x,y,z) = c$ —(1) represents a system of surfaces, wh 'c' is a parameter.

Now, we want to obtain a system of surfaces which cu each on eqⁿ (1) at right angles.

Then the direction ratios of the normal of the point $(x,y$ to eqⁿ (1) with which passes through the point $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

Let the surface $z = \phi(x,y)$ — (2) cuts each surface of eqⁿ at the right angles.

∴ The normal at the point (x,y,z) to eqⁿ(2) has direction ratio $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$

ie, $(p, q, -1)$

Since normals at (x,y,z) to eqⁿ (1) and eqⁿ(2) are at right angles we have

$$p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} = 0$$

$$\Rightarrow p\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$$

which is of the form $Pp + Qq = R$ —(3)

Conversely, any solⁿ of eqⁿ (3) is orthogonal to every surface of eq

1. Find the eqⁿ of the surface which intersects the surfaces of the system $z(x+y) = c(3z+1)$ orthogonally and passes through the circ $x^2+y^2=1,\ z=1$

Given eqⁿ :- $z(x+y) = c(3z+1)$ —(1) and $x^2+y^2=1,\ z=1$ —(A)

Now, we write eqⁿ (1) is of the form $f(x,y,z)=c$

From (1) we have $z(x+y) = c(3z+1)$

$$\Rightarrow c = \frac{z(x+y)}{3z+1}$$

$$\therefore f(x,y,z) = \frac{z(x+y)}{3z+1} \quad \text{—(B)}$$

Now $\frac{\partial f}{\partial x} = \frac{z}{3z+1}$

$\frac{\partial f}{\partial y} = \frac{z}{3z+1}$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z(x+y)}{3z+1}\right)$$

$$= (x+y)\frac{\partial}{\partial z}\left(\frac{z}{3z+1}\right)$$

$$= (x+y)\frac{d}{dz}\left[\frac{(3z+1)1 - z(3)}{(3z+1)^2}\right]$$

$$= (x+y)\left[\frac{3z+1-3z}{(3z+1)^2}\right]$$

$$\frac{\partial f}{\partial z} = \frac{(x+y)}{(3z+1)^2}$$

the required orthogonal surface is $P\frac{\partial f}{\partial x} + q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z}$

$$\Rightarrow P\frac{z}{3z+1} + q\frac{z}{3z+1} = \frac{x+y}{(3z+1)^2}$$

$$\Rightarrow pz(3z+1) + qz(3z+1) = x+y \quad -(2)$$

Eq^n (2) it is in form $Pp+Qq=R$

Here $P = z(3z+1)$

$Q = z(3z+1)$

$R = x+y$

The Lagrange's auxilary eq^n of (2) is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \quad -(3)$$

Taking first two terms

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)}$$

$dx = dy$

Integrate on both sides

$x = y + c_1$

$x - y = c_1 \quad -(4)$

choose $x, y, -z(3z+1)$ as multipliers of (3)

$$\frac{xdx + ydy - z(3z+1)dz}{xz(3z+1) + yz(3z+1) - (x+y)z(3z+1)}$$

$$\frac{xdx + ydy - 3z^2dz - zdz}{3xz^2 + xz + 3yz^2 + yz - 3z^2x - zx - 3z^2y - zy}$$

$\Rightarrow xdx + ydy - 3z^2dz - zdz$

Now integrate on both sides

$$\frac{x^2}{2} + \frac{y^2}{2} - 3\left(\frac{z^3}{3}\right) - \frac{z^2}{2} = c_2$$

$x^2 + y^2 - 2z^3 - z^2 = c_2 \quad -(5)$

Any surface which is orthogonal to (1) has eq^n of the form

$x^2 + y^2 - 2z^3 - z^2 = \phi(x, y)$

where 'ϕ' is an arbitrary function.

we want to find the real surface passing through the gi

Circle $x^2+y^2=1,\ z=1$

$\Rightarrow x^2+y^2=1$

$\Rightarrow y^2=1-x^2$ and $z=1$

Substitude this value in (5)

$x^2+y^2-2z^3-z^2=c_2$

$\cancel{x^2}+\cancel{1}-\cancel{x^2}-2(1)-\cancel{1}=c_2$

$c_2=-2$

From (5) we have $\phi(x,y)=-2$

$\therefore$ Required surface is $x^2+y^2-2z^3-z^2=-2 \Rightarrow x^2+y^2=2z^3+z^2-2$

→ Non-Linear partial Differential Eq^n's of First Order ←

The function $F(x,y,z,p,q)=0$ — (1) is called

In which the function F is non-linear in p & q

non-linear partial differential equation.

The partial differential eq^n of two parametric system is of the form $f(x,y,z,a,b)=0$ — (2) and x,y,z are variables

& z is dependent of 'x' and 'y'

Any partial differential eq^n of ① has a sol^n of the type (i)

They are various class of integrals of a partial differential eq^n of order I

a. Complete Integral:- consider a relation $\phi(x,y,z,a,b)=0$ —(1)

Differentiate (1) with respect to x and 'y'

with respect to 'x' $\Rightarrow \frac{\partial\phi}{\partial x}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial x}=0$

we know that $\frac{\partial z}{\partial x}=p$

$\Rightarrow \frac{\partial\phi}{\partial x}+p\frac{\partial\phi}{\partial z}=0$ — (2)

with respect to 'y' $\Rightarrow \frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial z}{\partial y}=0$

we know that $\frac{\partial z}{\partial y}=q$

$\Rightarrow \frac{\partial\phi}{\partial y}+\frac{\partial\phi}{\partial z}q=0$ — (4)

Since there are two arbitrary constants 'a' and 'b' connected by the above three equations, by eliminating these constants we get $f(x, y, z, p, q) = 0$ —(4) which is a first order partial differential eqn.

The integral of eqn (1) which has many arbitrary constants as they are independent variable is called the complete integral of eqn (4)

b, particular Integral:- A particular integral of eqn (4) is obtained by giving particular values to 'a' and 'b' in eqn (1)

c, Singular Integral:- we know that the locus of all the points whose co-ordinates along with the values of p and q satisfying eqn (4) represents the doubly infinite system of surfaces given by $\phi(x, y, z, a, b) = 0$ —(1)

(the system is doubly infinite means "Since there are two constants 'a' and 'b' and each of these can take an infinite number of values)

The envelop of all the surfaces given in eqn (1) is touched at each of these points by some point on the surface. The co-ordinates of any point on the envelop along with the values of 'p' and 'q' belongs to the envelop at that point and also satisfy the eqn $\frac{d\phi}{dy} + q\frac{d\phi}{dz} = 0$ —(2)

Hence the eqn of envelop is a soln of eqn (2)

∴ The envelop of the surface $\frac{d\phi}{dx} + p\frac{d\phi}{dz} = 0$ is obtained by eliminating the arbitrary constants 'a' and 'b' between the eqn's

$\phi(x, y, z, a, b) = 0$ and $\frac{d\phi}{dx} = 0, \frac{d\phi}{dy} = 0$ —(3)

The relation between xyz obtained is called the "Singular integral"

Note:- In general singular integral is distinct from the complete integral.

However, in exceptional cases it may be contained in the complete integral ie, singular integral may be obtained by giving

Particular values to the constants in the complete integ[ral]

D, General Integral:- Let us assume that in the eq^n $\phi(x, y, z, a, b) = 0$ — (1) one of the constant is a function of the other say $b = F(a)$ ie, $\phi(x, y, z, a, F(a)) = 0$ — (2) represents one of the families of the surface given by the system (1)

The eq^n of the envelop of the family of surfaces given by ~~two~~ (2) must also satisfy the eq^n $f(x, y, z, p, q) = 0$ — (3)

Again the eq^n so obtained will be distinct from that of the envelop of the surfaces and it is not a particular integr-al. It is known as general integral and is obtained by "Eliminating 'a' between $\phi(x, y, z, a, F(a)) = 0$ and $\frac{d\phi}{dx} = 0$

Note:- We can illustrate these kinds of sol^n with reference to a partial differential equation $z^2(1+p^2+q^2) = 1$ (Chapter II (Unit II) sol^n of the surface (Ex))

~~Sol: Given eq^n is~~

Ex:- Consider a P.D.E $z^2(1+p^2+q^2) = 1$ — (1)

We know that by previous problem (Chapter II (Unit II) Sol^n of the surface (Ex)) $(x-a)^2 + (y-b)^2 + z^2 = 1$ — (2) is a sol^n of (1) where 'a' and 'b' are arbitrary constants.

Since eq^n (2) is a sol^n of eq^n (1) and contains two arbitrary constants 'a' and 'b' so that the sol^n is a comple integral of (1)

Eq^n (2) is a one-parameter sub-system whose envelope is obtained by eliminating 'a' between $(x-a)^2 + (y-a)^2 + z^2 = 1$ and

$x + y - 2a = 0$ — (3)

$\Rightarrow \frac{x+y}{2} = a$

Substitude 'a' in (3)

$$\left(x - \left(\frac{x+y}{2}\right)\right)^2 + \left(y - \left(\frac{x+y}{2}\right)\right)^2 + z^2 = 1$$

$$\left(\frac{2x-x-y}{2}\right)^2 + \left(\frac{2y-x-y}{2}\right)^2 + z^2 = 1$$

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{-x+y}{2}\right)^2 + z^2 = 1$$

$$\frac{(x-y)^2}{4} + \frac{(-x+y)^2}{4} + z^2 = 1$$

$$\frac{2(x-y)^2}{4} + z^2 = 1$$

$$\frac{(x-y)^2}{2} + z^2 = 1$$

$\therefore \phi(x, y, z, a, F(a)) = (x-y)^2 + 2z^2 = 2$ —(5)

Now differentiate (5) with respect to 'x'

$$2(x-y) + 4z\frac{dz}{dx} = 0 \quad (\because p = \frac{dz}{dx})$$

$$2(x-y) + 4zp = 0$$

$$2zp = y - x \quad \text{—(6)}$$

Again differentiate (2) with respect to 'y'

$$-2(x-y) + 4z\frac{dz}{dy} = 0$$

$$-2(x-y) + 4zq = 0$$

$$2zq = -(y-x)$$

$$2zq = x - y \quad \text{—(7)}$$

$\therefore$ eq^n(5) is an integral surface of eq^n(1) and also is the general integral of (1).

$\therefore$ The envelop the two parameter system of eq^n(1) is obtained by eliminating 'a' and 'b' from eq^n(2)

$\therefore$ substitud $x - a = 0$, $y - b = 0$ in eq^n(2) we get $z^2 = 1$
$\Rightarrow z = \pm 1$

ie, the envelop consists of pair of planes $z = -1, 1$

→ Cauchy's Method of Characteristics ←

We shall now consider method of solving the non-linear partial differential eq^n $F(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}) = 0$ —(1)

In this section, we shall consider a method, due to cauchy which is based largely on geometrical ideas.

The plane passing through the point $P(x_0, y_0, z_0)$ with its normal parallel to the direction defined by the direction ratios $(p_0, q_0, -1)$ is uniquely specified by the set of numbers $D(x_0, y_0, z_0, p_0, q_0)$

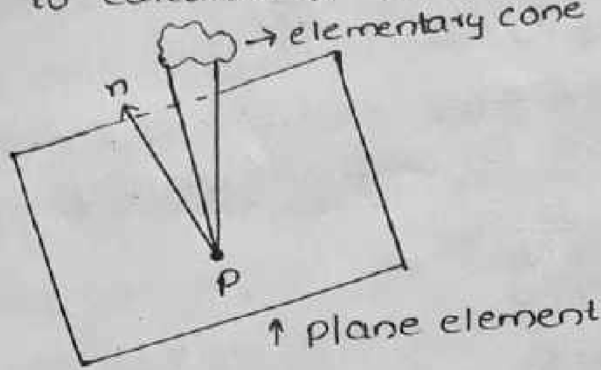
Conversely, any such set of five real numbers define a plane in three dimensional space. For this reason, a set of five numbers $D(x, y, z, p, q)$ is called a plane element of the space. In particular a plane element $(x_0, y_0, z_0, p_0, q_0)$ whose components satisfy an eq^n $F(x, y, z, p, q) = 0$ —(2) is called an "Integral element" of the eq^n (2) at the point (x_0, y_0, z_0)

It is theoretically possible to solve an eq^n of the type (2) to obtain an expression.

$$q = G(x, y, z, P) \quad \text{---(3)}$$

From which to calculate 'q' when x, y, z and p are known elen-

-ntary cone.



keeping x_n, y_0 and z_0 fixed and varying P, we obtain a set of plane elements $\{x_0, y_0, z_0, P, G(x_0, y_0, z_0, P)\}$ which dependent on the single parameter 'p'.

As 'p' varies, we obtain a set of plane elements of all o which pass through the point 'p' and which therefore envelop a cone vertex P, the cone so generated is called the elemen -tary cone of eq^n (2) at the point P (in figure)

Now consider a surface 'S' whose equation is $z = g(x, y)$ →(4)

If the function $g(x, y)$ and its first order partial differ derivatives with respect to x and y $g_x(x, y)$ and $g_y(x, y)$ are continuous in a region R of the xy-plane.

Then the tangent plane at each point of S determines a plane element of the type $\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\}$ — (5)

Eq^n (5) is called the "tangent element" of the surface 'S' at the point $\{x_0, y_0, g(x_0, y_0)\}$

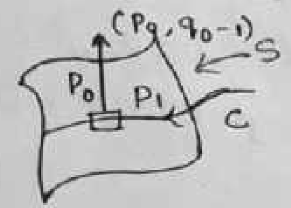
It may be proved in below theorem.

Theorem: Prove that a necessary and sufficient condition that a surface be an integral surface of a partial differential eqn is that at each point, its tangent element should touch the elementary cone of the eqn.

proof: we know that A curve 'c' with parametric equations

$x = x(t),\ y = y(t),\ z = z(t)$ — (1)

lie on the surface $z = g(x,y)$ if $z(t) = g\{x(t), y(t)\}$ for all values of 't' in the appropriate interval I.



If P_0 is a point in this case determined by the parameter t_0

Then the direction ratios of the tangent line $P_0 P_1$ are $\{x'(t_0), y'(t_0), z'(t_0)\}$ where $x'(t_0)$ denote the value of $\frac{dx}{dt}$ when $t = t_0$

This direction will be $\perp^{lar}$ to the direction $(p_0, q_0 - 1)$ if $z'(t_0) = p_0 x_0'(t_0) + q_0 y_0'(t_0)$

For this reason, we say that any set $\{x(t), y(t), z(t); p(t); q(t)\}$ $\rightarrow (2)$ of five real functions satisfying the condition.

$z'(t) = p(t)\, x'(t) + q(t)\, y'(t)$ — (3)

Eqn (3) defines (represents) a strip at the point (x,y,z) of the curve 'C'.

If such a strip exists, is also an integral element of the eqn $F(x,y,z,p,q) = 0$

Then we say that it is an integral strip of the eqn $F(x,y,z,p,q) = 0$

ie, the set of functions in eqn(2) is an integral strip of the eqn $F(x,y,z,p,q) = 0$

Also provided that they satisfy the condn (3) and further the condition

$F\{x(t), y(t), z(t), p(t), q(t)\} = 0 \ \forall\ [t]$ — (4)

At each point on the curve $x = x(t), y = y(t), z = z(t)$ touches a generator of the elementary cone

Then we say that a corresponding strip is a "characteristic strip".

Now we shall derive the equations determining a characteristic strip.

The point $(x+dx, y+dy, z+dz)$ in the tangent to the elementary cone at 'p' if

$$dz = p\,dx + q\,dy \quad \text{---}(5)$$

where p, q satisfy the relation $F(x,y,z,p,q)=0$

Now differentiate eq^n (5) with respect to 'p'

$$0 = dx + \frac{dq}{dp}dy \quad \text{---}(6)$$

Now differentiate the eq^n $F(x,y,z,p,q)=0$ with respect to 'p' we

$$\frac{dF}{dp} + \frac{dF}{dq} \cdot \frac{dq}{dp} = 0$$

$$\Rightarrow F_p + F_q \cdot \frac{dq}{dp} = 0 \quad \text{---}(7)$$

Now re-write the eq^n's (5), (6) & (7) we get

$$dz = p\,dx + q\,dy \quad \text{---}(5)$$

$$0 = dx + \frac{dq}{dp} dy \quad \text{---}(6)$$

$$F_p + F_q \cdot \frac{dq}{dp} = 0 \quad \text{---}(7)$$

Solve (5), (6) and (7) we get $\dfrac{dx}{F_p} = \dfrac{dy}{F_q} = \dfrac{dz}{pF_p + qF_q}$ ---(8)

So that along a characteristic strip, $x'(t), y'(t), z'(t)$ must be proportional to F_p, F_q, pF_p+qF_q respectively.

If we choose the parameter 't' in such a way that $x'(t) = F_p$, $y'(t) = F_q$ ---(9)

Then $z'(t) = pF_p + qF_q$ ---(10)

we know that along a characteristic strip 'p' is a function of 't'

so that $p'(t) = \dfrac{dp}{dx} x'(t) + \cancel{\dfrac{dp}{dq}y'(t)} \dfrac{\partial p}{\partial y} y'(t)$

$$p'(t) = \frac{dp}{dx} \cdot \frac{dF}{dp} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}$$

$$p'(t) = \frac{dp}{dx} \cdot \frac{dF}{dp} + \frac{dq}{dx}\frac{dF}{dq} \quad \text{---}(11) \qquad \left[\because \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}\right]$$

Now differentiate the eq^n $F(x,y,z,p,q)=0$ with respect to x, we get a generator of the elementary cone.

Then we say that a corresponding strip is a "characteristic strip".

Now we shall derive the equations determining a characteristic strip.

The point $(x+dx, y+dy, z+dz)$ in the tangent to the elementary cone at 'P' if

$$dz = p\,dx + q\,dy \quad -(5)$$

where p, q satisfy the relation $F(x,y,z,p,q)=0$

Now differentiate eq^n (5) with respect to 'p'

$$0 = dx + \frac{dq}{dp}dy \quad -(6)$$

Now differentiate the eq^n $F(x,y,z,p,q)=0$ with respect to 'p' we get

$$\frac{dF}{dp} + \frac{dF}{dq}\cdot\frac{dq}{dp} = 0$$

$$\Rightarrow F_p + F_q \cdot \frac{dq}{dp} = 0 \quad -(7)$$

Now re-write the eqn's (5), (6) & (7) we get

$$dz = p\,dx + q\,dy \quad -(5)$$

$$0 = dx + \frac{dq}{dp}dy \quad -(6)$$

$$F_p + F_q \cdot \frac{dq}{dp} = 0 \quad -(7)$$

Solve (5), (6) and (7) we get $\dfrac{dx}{F_p} = \dfrac{dy}{F_q} = \dfrac{dz}{pF_p + qF_q}$ —(8)

So that along a characteristic strip, $x'(t), y'(t), z'(t)$ must be proportional to F_p, F_q, pF_p+qF_q respectively.

If we choose the parameter 't' in such a way that $x'(t) = F_p,\ y'(t) = F_q$ —(9)

Then $z'(t) = pF_p + qF_q$ —(10)

We know that along a characteristic strip 'p' is a function of 't'

so that $p'(t) = \dfrac{dp}{dx}x'(t) + \dfrac{\partial p}{\partial y}y'(t)$

$$p'(t) = \frac{dp}{dx}\cdot\frac{dF}{dp} + \frac{\partial p}{\partial y}\frac{\partial F}{\partial q}$$

$$p'(t) = \frac{dp}{dx}\cdot\frac{dF}{dp} + \frac{dq}{dx}\frac{dF}{dq} \quad -(11) \qquad \left[\because \frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}\right]$$

Now differentiate the eq^n $F(x,y,z,p,q)=0$ with respect to x, we get

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial x}=0$$

We know that $\frac{\partial z}{\partial x}=p$

$$\Rightarrow \frac{\partial F}{\partial x}+\frac{\partial F}{\partial z}p+\frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=0$$

$$\Rightarrow \frac{\partial F}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial x}=-\frac{\partial F}{\partial x}-\frac{\partial F}{\partial z}p \quad \text{---(12)}$$

From eqn's (11) and (12) we have

$$p'(t)=-\frac{\partial F}{\partial x}-\frac{\partial F}{\partial z}p$$

$$\Rightarrow p'(t)=-(F_x+F_z p)$$

$$\Rightarrow p'(t)=-(F_x+pF_z) \quad \text{---(A)}$$

Similarly, along a characteristic strip 'q' is a function of 't' so that

$$q'(t)=\frac{\partial p}{\partial y}x'(t)+\frac{\partial q}{\partial y}y'(t)$$

$$q'(t)=\frac{\partial p}{\partial y}\frac{\partial F}{\partial p}+\frac{\partial q}{\partial y}\frac{\partial F}{\partial q}$$

$$q'(t)=\frac{\partial p}{\partial y}\frac{\partial F}{\partial p}+\frac{\partial q}{\partial y}\frac{\partial F}{\partial q}=0 \quad \text{---(13)}$$

Now, differentiate the eqn $F(x,y,z,p,q)=0$ with respect to 'y' we get

$$\frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y}+\frac{\partial F}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\cdot\frac{\partial q}{\partial y}=0$$

We know that $\frac{\partial z}{\partial y}=q$

$$\Rightarrow \frac{\partial F}{\partial y}+\frac{\partial F}{\partial z}q+\frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=0$$

$$\Rightarrow \frac{\partial F}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial F}{\partial q}\frac{\partial q}{\partial y}=-\frac{\partial F}{\partial y}-q\frac{\partial F}{\partial z} \quad \text{---(14)}$$

From (13) & (14) we have, $q'(t)=-\frac{\partial F}{\partial y}-\frac{\partial F}{\partial z}q$

$$\Rightarrow q'(t)=-(F_y+qF_z) \quad \text{---(B)}$$

Now collecting the eqn's 9, 10, A and B together, we see that we have the following system of the five ordinary differential eqn for the determination of the characteristic strip.

$$\left.\begin{array}{l} x'(t)=F_p,\ y'(t)=F_q,\ z'(t)=pF_p+qF_q \\ p'(t)=-F_x-pF_z \ \& \ q'(t)=-F_y-qF_z \end{array}\right\} \quad \text{---(15)}$$

These equations are known as characteristic equations of the partial differential eqn $F(x,y,z,p,q)=0$

Hence proved.

Note:- If the functions which appear in equation (15) which satisfies Libzitz condition there is a unique sol^n of the eq for each prescribed set of initial values of the variables

$\therefore$ The characteristic strip is uniquely determined by a initial elements $(x_0, y_0, z_0, p_0, q_0)$ and any initial value to

The main theorem about the characteristic strip is

Theorem:- Along every characteristic strip of the equation $F(x,y,$ $p,q)=0$ then the function $F(x,y,z,p,q)$ is a constant

Proof:- Along a characteristic strip, we have

$$\frac{d}{dt}\left[F\left(x(t), y(t), z(t), p(t), q(t)\right)\right] = F_x x'(t) + F_y y'(t) + F_z z'(t) + F_p p'(t) + F_q q'(t)$$

By above theorem, we have

$x'(t) = F_p$, $y'(t) = F_q$ & $z'(t) = pF_p + qF_q$

$p'(t) = -F_x - pF_z$; $q'(t) = -F_y - qF_z$

$$\therefore \frac{d}{dt}\left[F x(t), y(t), z(t), p(t), q(t)\right] = F_x F_p + F_y F_q + F_z(pF_p + qF_q) + F_p(-F_x - pF_z) + F_q(-F_y - qF_z)$$

$$= \cancel{F_x F_p} + \cancel{F_y F_q} + \cancel{F_z pF_p} + \cancel{qF_z F_q} - \cancel{F_x F_p} - \cancel{pF_pF_z} - \cancel{F_q F_y} - \cancel{qF_qF_z}$$

$$= 0$$

$$\therefore \frac{d}{dt}\left[F x(t), y(t), z(t), p(t), q(t)\right] = 0$$

$\therefore F(x, y, z, p, q) = k$ where 'k' is constant.

$\therefore$ The function $F(x, y, z, p, q) = 0$ is a constant function.

Hence proved.

Theorem:- If a characteristic strip contains atleast one integral element of $F(x,y,z,p,q)=0$ it is an integral strip of the eq^n $F(x, y, z, z_x, z_y) = 0$

Proof:- Now, we are in a position to solve cauchy's problem.

Suppose we wish to find the sol^n of the partial differential eq^n $F\left(x, y, z, \frac{dz}{dx}, \frac{dz}{dy}\right) = 0$ —(1) which passess through a curve Γ whose freedom eq^n's are

$x = \theta(v)$, $y = \phi(v)$, $z = \psi(v)$ — (2)

Then in the sol^n $x = x(p_0, q_0, x_0, y_0, z_0, t_0, t)$ —(3) of the characteristic equations

$x'(t) = F_p,\ y'(t) = F_q,\ z'(t) = pF_p + qF_q$

$p'(t) = -F_x - pF_z,\ q'(t) = -F_y - qF_z$

Then we take $x_0 = \theta(\upsilon),\ y_0 = \phi(\upsilon),\ z_0 = \psi(\upsilon)$ as the initial values of x, y, z

The corresponding initial values of p_0, q_0 are determined by the relations.

$$\psi'(\upsilon) = p_0\theta'(\upsilon) + q_0\phi'(\upsilon)$$

Since $F(x_0, y_0, z_0, p_0, q_0) = 0$

Substitude $x_0 = \theta(\upsilon),\ y_0 = \phi(\upsilon),\ z_0 = \psi(\upsilon)$

$\Rightarrow F(\theta(\upsilon), \phi(\upsilon), \psi(\upsilon), p_0, q_0) = 0$

If we Substitude these values of $(x_0, y_0, z_0, p_0, q_0)$ and the appropirate value of t_0 in eqn (2) we find that x, y, z can be expressed in the terms of two parameters t, υ to give $x = x_1(\upsilon, t)$, $y = y_1(\upsilon, t),\ z = z_1(\upsilon, t)$ —(4)

Now, eliminating υ, t from these three eqn's we get a relation $\varphi(x, y, z) = 0$

which is the eqn of the integral surface of the eqn (1) through the curve Π.

we shall illustrate this procedure by example.

Example:- Find the soln of the eqn $z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$ which passes through the x-axis.

Sol:- we know that eqn of P.D.E is $f(x, y, z, p, q) = 0$

Let us take initial values for the eqn $F(x_0, y_0, z_0, p_0, q_0, t_0) = 0$ are $x_0 = \upsilon,\ y_0 = 0,\ z_0 = 0,\ p_0 = 0,\ q_0 = 2\upsilon,\ t_0 = 0$ —(1)

we know that the characteristic eqn is :-

$x'(t) = F_p,\ y'(t) = F_q;\ z'(t) = pF_p + qF_q$

$p'(t) = -F_x - pF_z;\ q'(t) = -F_y - qF_z$

given eqn is :- $z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$

Now, let us find the characteristic eqn's

$$x'(t) = \frac{dx}{dt} = p + q - y$$

$$y'(t) = \frac{dy}{dt} = p + q - x$$

$$z'(t) = \frac{dz}{dt} = p(p + q - y) + q(p + q - x)$$

$$\left.\begin{aligned} p'(t) &= \frac{dp}{dt} = p+q-y \\ q'(t) &= \frac{dq}{dt} = p+q-x \end{aligned}\right\} \text{ --- (A)}$$

Since from (1), we have $x_0 = v$, $q_0 = 2v$

Let $x = v+p$, $y = q-2v$

Now differentiating $p+q-x$ & $p+q-y$ with respect to 't' part

$$\frac{d}{dt}(p+q-x) = p+q-x \quad \& \quad \frac{d}{dt}(p+q-y) = p+q-y$$

Since $x = v+p$ and $y = q-2v$

Let $p+q-x = v_e t$ — (1) ve^t

$p+q-y = 2v_e t$ — (2) $2ve^t$

Since $x = v+p \Rightarrow p = x-v$

Substitude 'p' in (1)

$(1) \Rightarrow \cancel{x} - v + q - \cancel{x} = v_e t$

$\Rightarrow q = v_e t + v$ $\quad q = ve^t + v = v(e^t+1)$

Now substitude q value in $y = q-2v$

$y = v_e t + v - 2v$

$\therefore y = v_e t - v$

Now substitude y and q in eqⁿ (2)

$p + \cancel{v_e t} + \cancel{v} - \cancel{v_e t} + \cancel{v} = 2v_e t$

$p = 2v_e t - 2v$

$p = 2v(e^t - 1)$

Substitude p, q in (1)

$(1) \Rightarrow p+q-x = v_e t$

$2v(e^t-1) + v_e t + v - x = v_e t$

$2v_e t - 2v + \cancel{v_e t} + v - x = \cancel{v_e t}$

$x = 2v_e t - v$

$x = v(2e^t - 1)$

$$\left.\begin{aligned} &\therefore x = v(2e^t-1) \text{ and } y = v(e^t-1) \\ &p = 2v(e^t-1) \text{ and } q = v_e t + v = v(e^t+1) \end{aligned}\right\} \text{ — (B)}$$

Substitude these values in third one of the characteristic eqⁿ

from (A) $\therefore z'(t) = \dfrac{dz}{dt}$

$= p(p+q-y) + q(p+q-x)$

$= p^2 + pq - py + qp + q^2 - xq$

$\Rightarrow z'(t) = p^2 + q^2 + 2pq - py - xq$

Substitude p, q, x, y in above eqⁿ

$z'(t) = (2\upsilon(e^t-1))^2 + (\upsilon e^t+\upsilon)^2 + 2(2\upsilon(e^t-1))(\upsilon e^t+\upsilon) - 2\upsilon(e^t-1)\upsilon(e^t-1) - (\upsilon)(2e^t-1)(\upsilon e^t+\upsilon)$

$= 4\upsilon^2(e^t-1)^2 + \upsilon^2(e^t+1)^2 + 4\upsilon^2(e^t-1)(e^t+1) - 2\upsilon^2(e^t-1)^2 - \upsilon^2(e^t+1)(2e^t-1)$

$= 4\upsilon^2(e^{2t}+1-2e^t) + \upsilon^2(e^{2t}+1+2e^t) + 4\upsilon^2(e^{2t}-1) - 2\upsilon^2(e^{2t}+1-2e^t) - \upsilon^2(2e^{2t}+e^t-1)$

$= 4\upsilon^2 e^{2t} + \cancel{4\upsilon^2} - 8e^t\upsilon^2 + \upsilon^2 e^{2t} + \cancel{\upsilon^2} + 2\upsilon^2 e^t + 4\upsilon^2 \cancel{e^{2t}} - \cancel{4\upsilon^2} - 2\cancel{\upsilon^2} e^{2t} - 2\cancel{\upsilon^2} + 4\upsilon^2 e^t + \cancel{\upsilon^2} \cancel{2e^{2t}} - \upsilon^2 e^t + \cancel{\upsilon^2}$

$z'(t) = 5\upsilon^2 e^{2t} - 3\upsilon^2 e^t$

Now integrate on both sides

$z = 5\upsilon^2 \dfrac{e^{2t}}{2} - 3\upsilon^2 e^t$

$z = \dfrac{5}{2}\upsilon^2(e^{2t}-1) - 3\upsilon^2(e^t-1)$

By estimation, we have $z = \dfrac{5}{2}\upsilon^2(e^{2t}-1) - 3\upsilon^2(e^t-1)$ —— (3)

In eqⁿ (B) we have the valuep of x, y, z, p, q

consider, $\dfrac{y-x}{2y-x} = \dfrac{\upsilon(e^t-1) - \upsilon(2e^t-1)}{2\upsilon(e^t-1) - \upsilon(2e^t-1)}$

$= \dfrac{\upsilon e^t - \upsilon - 2\upsilon e^t + \upsilon}{2\cancel{\upsilon}e^t - 2\upsilon - 2\cancel{\upsilon e^t} + \upsilon}$

$= \dfrac{\cancel{-}\cancel{\upsilon}e^t}{\cancel{-}\cancel{\upsilon}} = e^t$

$\therefore \dfrac{y-x}{2y-x} = e^t$

From (B) $\Rightarrow x - 2y = \upsilon(2e^t-1) - 2\upsilon(e^t-1)$

$= 2\cancel{\upsilon}e^t - \upsilon - 2\cancel{\upsilon}e^t + 2\upsilon$

$= \upsilon$

$\therefore x - 2y = \upsilon$

Substitude υ, e^t values in (3)

$z = \dfrac{5}{2}\upsilon^2(e^{2t}-1) - 3\upsilon^2(e^t-1)$

$z = \dfrac{5}{2}(x-2y)^2\left[\left[\dfrac{y-x}{2y-x}\right]^2 - 1\right] - 3(x-2y)^2\left[\dfrac{y-x}{2y-x} - 1\right]$

$= \dfrac{5}{2}(x-2y)^2\left[\dfrac{(x-y)^2 - (x-2y)^2}{(x-2y)^2}\right] - 3(x-2y)^2\left[\dfrac{x-y-x+2y}{x-2y}\right]$

$= \dfrac{5}{2}\left[(x-y)^2 - (x-2y)^2\right] - 3((x-2y)y)$

$= \dfrac{5}{2}\left[x^2+y^2-2xy - x^2 - 4y^2 + 4xy\right] - 3(xy-2y^2)$

$$= \frac{5}{2}(-3y^2 + 2xy) - 3(xy - 2y^2)$$

$$= \frac{1}{2}(-15y^2 + 10xy) - 3xy + 6y^2$$

$$= \frac{-15y^2 + 10xy - 6xy + 12y^2}{2}$$

$$z = \frac{4xy - 3y^2}{2}$$

$$\therefore z = \frac{1}{2} y(4x - 3y)$$

Hence proved.

→ Compatible System of First Order Equations ←

Compatible:- The equations $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$ are compatible. If, $J=\frac{\partial(f,g)}{\partial(p,q)}\neq 0$ and where $p=\phi(x,y,z)$ and $q=\psi(x,y,z)$ are obtained by solving 'f' and 'g' then we get $dz=\phi(x,y,z)\,dx+\psi(x,y,z)\,dy$ is integrable.

Theorem:- A necessary and sufficient condition for the integrability of $dz=\phi(x,y,z)\,dx+\psi(x,y,z)\,dy$ is $[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0$ where $p=\phi(x,y,z)$ and $q=\psi(x,y,z)$

Proof:- Given equation is $\phi(x,y,z)\,dx+\psi(x,y,z)\,dy-dz=0$ —(1)

Eq^n (1) is of the form of pfaffin differential eq^n i.e,

$Pdx+Qdy+Rdz=0$

we know that a pfaffin differential equation is integrable $\Leftrightarrow$

$\bar{X}\cdot \text{curl}\,\bar{X}=0$

From eq^n (1), we have $P=\phi(x,y,z)$; $Q=\psi(x,y,z)$; $R=-1$

we know that $\bar{X}=(P,Q,R)\,(i,j,k)$

$$\Rightarrow \bar{X}=(\phi(x,y,z)i+\psi(x,y,z)j-1k)$$

we know that $\text{curl}\,\bar{X}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & R\end{vmatrix}=\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi & \psi & -1\end{vmatrix}$ ($\because$ Let $\phi(x,y,z)=\phi$ and $\psi(x,y,z)=\psi$)

$$= i\left[\frac{-\partial}{\partial y}(1)-\frac{\partial\psi}{\partial z}\right]-j\left[\frac{-\partial}{\partial x}(1)-\frac{\partial\phi}{\partial z}\right]+k\left[\frac{\partial\psi}{\partial x}-\frac{\partial\phi}{\partial y}\right]$$

$\text{curl}\,\bar{X}=-i\psi_z+j\phi_z+k(\psi_x-\phi_y)$

we know that $\bar{X}\cdot\text{curl}\,\bar{X}=0$

Now $\bar{X}\cdot\text{curl}\,\bar{X}=(\phi,\psi,-1)(-\psi_z,\phi_z,\psi_x-\phi_y)=0$

$\Rightarrow -\phi\psi_z+\psi\phi_z-\psi_x+\phi_y=0$

$\Rightarrow \psi\phi_z+\phi_y=\phi\psi_z+\psi_x$ —(2)

Now, consider the differential eq^n's $f(x,y,z,p,q)=0$ —(3) &

$g(x,y,z,p,q)=0$ —(4)

Hence $p=\phi(x,y,z)$, $q=\psi(x,y,z)$

Now, differentiate eq^n (3) with respect to x partially

$\Rightarrow \frac{\partial f}{\partial x}\frac{\partial x}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial x}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial x}=0$

$\Rightarrow f_x+0+0+f_p\,p_x+f_q\,q_x=0$

$\Rightarrow f_x + f_p P_x + f_q q_x = 0$

Since $p = \phi(x,y,z)$ & $q = \varphi(x,y,z)$

$\Rightarrow f_x + f_\phi \phi_x + f_\varphi \varphi_x = 0$ — (5) 6

Now differentiate eqn (3) with respect to 'z' partially then we get

$\Rightarrow f_z + f_\phi \phi_z + f_\varphi \varphi_z = 0$ — (6)

Now multiply eqn 6 with ϕ

$\Rightarrow \phi f_z + \phi f_\phi \phi_z + \phi f_\varphi \varphi_z = 0$ — (7)

Adding (5) and (7) $f_x + f_\phi \phi_x + f_\varphi \varphi_x + \phi f_z + \phi f_\phi \phi_z + \phi f_\varphi \varphi_z = 0$

$\Rightarrow f_x + \phi f_z + f_\phi (\phi_x + \phi \phi_z] + f_\varphi [\varphi_x + \phi \varphi_z] = 0$ — (8)

Now differentiate eqn (4) with respect to 'x' partially

$\Rightarrow \frac{\partial g}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial g}{\partial z}\cdot\frac{\partial z}{\partial x} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} = 0$

$\Rightarrow g_x + 0 + 0 + g_p P_x + g_q q_x = 0$

$\Rightarrow g_x + g_p P_x + g_q q_x = 0$

Since $p = \phi(x,y,z)$ & $q = \varphi(x,y,z)$

$\Rightarrow g_x + g_\phi \phi_x + g_\varphi \varphi_x = 0$ — (9)

Now differentiate eqn (4) with respect to 'z' partially then we get

$\Rightarrow g_z + g_\phi \phi_z + g_\varphi \varphi_z = 0$ — (10)

Now multiply eqn 10 with ϕ

$\Rightarrow \phi g_z + \phi g_\phi \phi_z + \phi g_\varphi \varphi_z = 0$ — (11)

Adding 9 & (11) $g_x + g_\phi \phi_x + g_\varphi \varphi_x + \phi g_z + \phi g_\phi \phi_z + \phi g_\varphi \varphi_z = 0$

$\Rightarrow g_x + \phi g_z + g_\phi(\phi_x + \phi \phi_z) + g_\varphi(\varphi_x + \phi \varphi_z) = 0$ — (12)

From eqn's 8 and 12 we have

$f_x + \phi f_z + f_\phi(\phi_x + \phi \phi_z) + f_\varphi(\varphi_x + \phi \varphi_z) = 0$

$g_x + \phi g_z + g_\phi(\phi_x + \phi \phi_z) + g_\varphi(\varphi_x + \phi \varphi_z) = 0.$

$f_x + \phi f_z + f_\varphi(\varphi_x + \phi \varphi_z) = -f_\phi(\phi_x + \phi \phi_z)$

$g_x + \phi g_z + g_\varphi(\varphi_x + \phi \varphi_z) = -g_\phi(\phi_x + \phi \phi_x)$

$\Rightarrow \frac{f_x + \phi f_z + f_\varphi(\varphi_x + \phi \varphi_z)}{g_x + \phi g_z + g_\varphi(\varphi_x + \phi \varphi_z)} = \frac{\cancel{-} f_\phi(\phi_x + \cancel{\phi}\phi_z)}{\cancel{-} g_\phi(\phi_x + \cancel{\phi}\phi_z)}$

$\Rightarrow g_\phi(f_x + \phi f_z + f_\varphi(\varphi_x + \phi\varphi_z)) = f_\phi(g_x + \phi g_z + g_\varphi(\varphi_x + \phi\varphi_z))$

$\Rightarrow g_\phi f_x + \phi g_\phi f_z + g_\phi f_\varphi(\varphi_x + \phi\varphi_z) = f_\phi g_x + \phi f_\phi g_z + f_\phi g_\varphi(\varphi_x + \phi\varphi_z)$

$\Rightarrow g_\phi f_\varphi(\varphi_x + \phi\varphi_z) - f_\phi g_\varphi(\varphi_x + \phi\varphi_z) = f_\phi g_x + \phi f_\phi g_z - g_\phi f_x - \phi g_\phi f_z$

$\Rightarrow (\varphi_x + \phi\varphi_z)(g_\phi f_\varphi - f_\phi g_\varphi) = f_\phi g_x - g_\phi f_x + \phi(f_\phi g_z - g_\phi f_z)$

we know that $\dfrac{\partial(u,v)}{\partial(x,y)} = u_x v_y - v_x u_y$

$$\Rightarrow \frac{\partial(g,f)}{\partial(\phi,\varphi)}(\varphi_x + \phi\varphi_z) = \frac{\partial(f,g)}{\partial(\phi,x)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)} \quad \text{—— (A)}$$

let $J = \dfrac{\partial(f,g)}{\partial(\phi,\varphi)} \Rightarrow -J = +\dfrac{\partial(g,f)}{\partial(\phi,\varphi)}$

substitude J value in (A)

$$\Rightarrow -J(\varphi_x + \phi\varphi_z) = \frac{\partial(f,g)}{\partial(\phi,x)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)}$$

$$\Rightarrow \varphi_x + \phi\varphi_z = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(\phi,x)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)}\right] \text{—— (13)}$$

Similarly, by differentiating (3) & (4) with respect to y & z, then we get

$$\Rightarrow (\phi_y + \varphi\phi_z) = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,\varphi)} + \varphi\frac{\partial(f,g)}{\partial(z,\varphi)}\right] \text{—— (14)}$$

From eqn (2) we have, $\varphi\phi_z + \phi_y = \phi\varphi_z + \varphi_x$

$$\Rightarrow -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(\phi,x)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)}\right] = -\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,\varphi)} + \varphi\frac{\partial(f,g)}{\partial(z,\varphi)}\right]$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(\phi,x)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)} - \frac{\partial(f,g)}{\partial(y,\varphi)} - \varphi\frac{\partial(f,g)}{\partial(z,\varphi)} = 0$$

Since $p = \phi(x,y,z)$ and $q = \varphi(x,y,z)$

Now replace 'ϕ' and 'φ' with 'p' and 'q' we get

$$\frac{\partial(f,g)}{\partial(p,x)} + p\frac{\partial(f,g)}{\partial(p,z)} - \frac{\partial(f,g)}{\partial(y,q)} - q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$$\Rightarrow -\frac{\partial(f,g)}{\partial(x,p)} - p\frac{\partial(f,g)}{\partial(z,p)} - \frac{\partial(f,g)}{\partial(y,q)} - q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$\Rightarrow [f,g] = 0$

$\therefore [f,g] = 0$

Hence proved.

Note:- A solⁿ of $dz = \phi(x,y,z)\,dx + \psi(x,y,z)\,dy$ is of the form $F(x,y,z,c)=0$ where c is an arbitrary parameter.

Hence we assert that if the equations $f(x,y,z,p,q)=0$ & $g(x,y,z,p,q)=0$ are compatiable then they have a non-parameter family of common solutions.

Problems:-

1. Show that the eqⁿ's $xp-yq=x$, $x^2p+q=xz$ are compatiable and find the solⁿ.

Sol:- Given:- $f(x,y,z,p,q) = xp-yq-x=0$ —(1)

$g(x,y,z,p,q) = x^2p+q-xz=0$ —(2)

To prove (1) & (2) are compatiable we have to prove that $[f,g]=0$

ie, $\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$ —(3)

From (3) consider $\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix}$

From eqⁿ (1) we have, $\frac{\partial f}{\partial x} = p-1$, $\frac{\partial f}{\partial p} = x$

From eqⁿ (2) we have, $\frac{\partial g}{\partial x} = 2xp-z$, $\frac{\partial g}{\partial p} = x^2$

$\therefore \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = \frac{\partial(f,g)}{\partial(x,p)}$

$\Rightarrow \frac{\partial(f,g)}{\partial(x,p)} = x^2(p-1) - x(2xp-z)$ —(4)

consider, $\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix}$

from (1) we have, $\frac{\partial f}{\partial z} = 0$, $\frac{\partial f}{\partial p} = x$

From (2) we have, $\frac{\partial g}{\partial z} = -x$, $\frac{\partial g}{\partial p} = x^2$

$\Rightarrow \frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix}$

$\therefore \frac{\partial(f,g)}{\partial(z,p)} = x^2$ —(5)

consider, $\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix}$

From (1) we have, $\frac{\partial f}{\partial y} = -q$, $\frac{\partial f}{\partial q} = -y$

From (2) we have $\frac{\partial g}{\partial y} = 0$ $\frac{\partial g}{\partial q} = 1$

$$\Rightarrow \frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix}$$

$$\therefore \frac{\partial(f,g)}{\partial(y,q)} = -q \quad \text{---}(6)$$

consider, $\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix}$

From (1) we have, $\frac{\partial f}{\partial z} = 0 \quad \frac{\partial f}{\partial q} = -y$

From (2) we have, $\frac{\partial g}{\partial z} = -x \quad \frac{\partial g}{\partial q} = 1$

$$\Rightarrow \frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix}$$

$$\therefore \frac{\partial(f,g)}{\partial(z,q)} = -xy \quad \text{---}(7)$$

$(3) \Rightarrow x^2(p-1) - x(2xp) + xz + px^2 - q - qxy \neq 0$

$\Rightarrow \cancel{x^2p - x^2 - 2x^2p + xz + px^2 - q - xy = 0}$

$\Rightarrow x^2\cancel{p} - x^2 - 2x^2\cancel{p} + xz + p\cancel{x^2} - q - qxy \neq 0$

$\Rightarrow x^2 - q - qxy + xz \neq 0$

$\Rightarrow -\cancel{q}(\ -(q - xz) - x^2 - qxy \neq 0$

From (2) $\Rightarrow x^2p + q - xz = 0$

$\Rightarrow q - xz = -x^2p$

$\Rightarrow -(-x^2p) - x^2 - qxy$

$\Rightarrow x^2p - x^2 - qxy$

$\Rightarrow x(xp - x - qy)$ ($\because xp - yq - x = 0$ from (1))

$\Rightarrow x(0)$

$= 0$

$$\therefore \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

$\therefore$ Given eqn's are compatiable.

Now we find soln of (1) & (2) by eliminating 'p' & 'q'

From (1) we have, $xp - yq = x$

$yq = xp - x$

$q = \frac{xp - x}{y}$

Substitude 'q' value in (2) $x^2p + q - xz = 0$

$x^2p - xz + \frac{xp - x}{y} = 0$

$x^2py - xzy + xp - x = 0$

$\therefore xpy - zy + p - 1 = 0$

$$P(xy+1) = 1+zy$$

$$P = \frac{1+yz}{xy+1}$$

Substitude 'p' in $q = \frac{xp-x}{y}$

$$= \frac{x\left(\frac{1+yz}{xy+1}\right)-x}{y}$$

$$= \frac{x(1+yz)-x(xy+1)}{y(xy+1)}$$

$$= \frac{\cancel{x}+xyz-x^2y-\cancel{x}}{y(xy+1)} = \frac{x\cancel{y}(z-x)}{\cancel{y}(xy+1)}$$

$$q = \frac{x(z-x)}{xy+1}$$

Substitude p & q in $dz = pdx+qdy$ we have

$$\Rightarrow \left(\frac{1+yz}{xy+1}\right)dx + \left(\frac{x(z-x)}{xy+1}\right)dy = dz$$

$$\Rightarrow \frac{(1+yz)dx + x(z-x)dy}{xy+1} = dz$$

$$dx(1+yz) + x(z-x)\,dy = (xy+1)dz$$

$$dx - x^2dy + z(ydx+xdy) = (1+xy)dz$$

$$dx - x^2\,dy + zd(xy) = (1+xy)dz$$

$$(1+xy)dz - zd(xy) = dx - x^2dy$$

Divide with $(1+xy)^2$

$$\Rightarrow \frac{(1+xy)dz - zd(xy)}{(1+xy)^2} = \frac{dx-x^2dy}{(1+xy)^2}$$

$$d\left(\frac{z}{1+xy}\right) = \frac{-x^2\left(\frac{-1}{x^2}dx+dy\right)}{\left(x\left(y+\frac{1}{x}\right)\right)^2}$$

$$= \frac{-\cancel{x^2}\left(-\frac{1}{x^2}dx+dy\right)}{\cancel{x^2}(y+1/x)^2}$$

$$= -\frac{\left(\frac{-1}{x^2}dx+dy\right)}{\left(\frac{1}{x}+y\right)^2} = -\frac{\left(\frac{-1}{x^2}dx+dy\right)}{\left(\frac{1}{x}+y\right)^2}$$

$$d\left(\frac{z}{1+xy}\right) = \frac{-d(y+1/x)}{\left(\frac{1}{x}+y\right)^2} \quad \left(\because d\left(\frac{1}{x}\right) = \frac{-1}{x^2}\right)$$

Integrate on both sides, then we get

$$\frac{z}{1+xy} = \frac{1}{y+1/x} + c$$

$$\frac{z}{1+xy} = \frac{x}{x(y)+1} + c \quad \Rightarrow \frac{z-x}{1+xy} = c$$

$\therefore z-x = c(1+xy)$ is the required soln.

If $u_1 = \frac{\partial u}{\partial x}$, $u_2 = \frac{\partial u}{\partial y}$, $u_3 = \frac{\partial u}{\partial z}$ then show that the equations $f(x, y, z, u_1, u_2, u_3) = 0$ and $g(x, y, z, u_1, u_2, u_3) = 0$ are compatiable if

$$\frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(z,u_3)} = 0$$

Sol:- we know that two first order partial differential eq^{n}s

$$\left.\begin{array}{l} f(x, y, z, p, q) = 0 \\ g(x, y, z, p, q) = 0 \end{array}\right\} \text{—(1)}$$ are compatible if $[f, g] = 0$

we know that $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$ —(A)

since given P.D.E's are $$\left.\begin{array}{l} f(x, y, z, u_1, u_2, u_3) = 0 \\ g(x, y, z, u_1, u_2, u_3) = 0 \end{array}\right\} \text{—(2)}$$

From (1) and (2) we have $p = u_1$ & $q = u_2$

And also treating 'z' as a constant, substitude $z = u$

substitude $p = u_1$; $q = u_2$ & $z = u$ in (A)

$$\frac{\partial(f,g)}{\partial(x,u_1)} + p\frac{\partial(f,g)}{\partial(u,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + q\frac{\partial(f,g)}{\partial(u,u_2)} = 0 \text{ — (3)}$$

since 'f' & 'g' does not contain 'u'

$\Rightarrow \frac{\partial f}{\partial u} = 0$ & $\frac{\partial g}{\partial u} = 0$

we know that if $\frac{\partial(f,g)}{\partial(x,u_1)} + u_1\frac{\partial(f,g)}{\partial(u,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + u_2\frac{\partial(f,g)}{\partial(u,u_2)} = 0$ —(3)

Since f and g does not contain u $\Rightarrow \frac{\partial f}{\partial u} = 0$ & $\frac{\partial g}{\partial u} = 0$

we know that $\frac{\partial(f,g)}{\partial(u,u_1)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial u_1} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u_1} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial f}{\partial u_1} \\ 0 & \frac{\partial g}{\partial u_1} \end{vmatrix}$

$$\therefore \frac{\partial(f,g)}{\partial(u,u_1)} = 0 \text{ — (4)}$$

Now we have $\frac{\partial(f,g)}{\partial(u,u_2)} = \begin{vmatrix} \frac{\partial f}{\partial u} & \frac{\partial f}{\partial u_2} \\ \frac{\partial g}{\partial u} & \frac{\partial g}{\partial u_2} \end{vmatrix} = \begin{vmatrix} 0 & \frac{\partial f}{\partial u_2} \\ 0 & \frac{\partial g}{\partial u_2} \end{vmatrix}$

$$\therefore \frac{\partial(f,g)}{\partial(u,u_2)} = 0 \text{ — (5)}$$

Now substitude 4 & 5 in (3)

$(3) \Rightarrow \frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} = 0$ — (6)

Now substitude $p = u_1$, $q = u_3$ in (A) we get

$$\frac{\partial(f,g)}{\partial(x,u_1)} + u_1\frac{\partial(f,g)}{\partial(z,u_1)} + \frac{\partial(f,g)}{\partial(y,u_3)} + u_3\frac{\partial(f,g)}{\partial(z,u_3)} = 0 \text{ — (7)}$$

Since we have $\frac{\partial f}{\partial u} = 0$ & $\frac{\partial g}{\partial u} = 0$ when 'z' is a constant

In this case, also we have $\frac{\partial(f,g)}{\partial(z,u_1)} = 0$ & $\frac{\partial(f,g)}{\partial(z,u_3)} = 0$

Substitude these values in (7) we get

$$\frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_3)} = 0$$

Already we have z is a constant

Substitude $y=z$ in above eqn, we get

$$\frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(z,u_3)} = 0 \quad \text{---}(8)$$

Substitude $p = u_2, q = u_3$ in (A)

$$\frac{\partial(f,g)}{\partial(x,u_2)} + u_2\frac{\partial(f,g)}{\partial(z,u_2)} + \frac{\partial(f,g)}{\partial(y,u_3)} + u_3\frac{\partial(f,g)}{\partial(z,u_3)} = 0 \quad \text{---}(9)$$

we have 'z' is a constant, $\frac{\partial f}{\partial z} = 0$ & $\frac{\partial g}{\partial z} = 0$

Then above eqn becomes, $\frac{\partial(f,g)}{\partial(z,u_2)} = 0$ & $\frac{\partial(f,g)}{\partial(z,u_3)} = 0$

$$(9) \Rightarrow \frac{\partial(f,g)}{\partial(x,u_2)} + \frac{\partial(f,g)}{\partial(y,u_3)} = 0$$

Substitude $x=y$ & $y=z$

$$\Rightarrow \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(z,u_3)} = 0 \quad \text{---}(10)$$

From (6), (8) & (10) we have and apply addition on it

$$\frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(z,u_3)} + \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(z,u_3)} = 0$$

$$\Rightarrow 2\left[\frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(z,u_3)}\right] = 0$$

$$\therefore \frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(z,u_3)} = 0$$

$\therefore$ the given equations $f(x,y,z,u_1,u_2,u_3) = 0$ & $g(x,y,z,u_1,u_2,u_3) = 0$

are compatiable if $\frac{\partial(f,g)}{\partial(x,u_1)} + \frac{\partial(f,g)}{\partial(y,u_2)} + \frac{\partial(f,g)}{\partial(z,u_3)} = 0$

3. Show that the eq^n's $f(x,y,z,p,q)=0$, $g(x,y,z,p,q)=0$ are compatible if $\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$ verify that the eq^n's $p=P(x,y)$ & $q=Q(x,y)$ are compatiable if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

Sol: we know that the equations $f(x,y,z,p,q)=0$ & $g(x,y,z,p,q)=0$ $\rightarrow(1)$

are compatible if $[f,g]=\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0$ —(2)

Claim: 1, the equations $p=P(x,y)$ & $q=Q(x,y)$ are compatible if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$

2, the equations $f(x,y,p,q)=0$ and $g(x,y,p,q)=0$ are compatible if $\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$

To prove 2 :- comparing the equations $f(x,y,p,q)=0$ and $g(x,y,p,q)=0$ with ①

we find that 'z' is absent in given eq^n's and so $\frac{\partial f}{\partial z}=0$ & $\frac{\partial g}{\partial z}=0$

from eq^n(2) consider, $\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial p}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p}\end{vmatrix}$ ($\because z$ is absent)

$$=\begin{vmatrix}0 & \frac{\partial f}{\partial p}\\ 0 & \frac{\partial g}{\partial p}\end{vmatrix}=0$$

$$\therefore \frac{\partial(f,g)}{\partial(z,p)}=0$$

Now $\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\frac{\partial f}{\partial z} & \frac{\partial f}{\partial q}\\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q}\end{vmatrix}$

$$=\begin{vmatrix}0 & \frac{\partial f}{\partial q}\\ 0 & \frac{\partial g}{\partial q}\end{vmatrix}=0$$

$$\therefore \frac{\partial(f,g)}{\partial(z,q)}=0$$

Substitude these values in (2) we get, $\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$

$\therefore$ The eq^n's $f(x,y,p,q)=0$ & $g(x,y,p,q)=0$ are compatible if

$\frac{\partial(f,g)}{\partial(x,p)}+\frac{\partial(f,g)}{\partial(y,q)}=0$ —(3)

To Prove 1 :- Since $p=P(x,y)$ and $q=Q(x,y)$

Let $f=P(x,y)-p$, $g=Q(x,y)-q$ —(4)

compare these eq^n's with eq^n(1)

we find that z,q are z,q are absent in f and z,p are absent in g.

So $\frac{\partial f}{\partial z}=0$, $\frac{\partial f}{\partial q}=0$, $\frac{\partial g}{\partial z}=0$, $\frac{\partial g}{\partial p}=0$ — (5)

From eq^n (2) we find all the values

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{\partial P}{\partial x} & -1 \\ \frac{\partial Q}{\partial x} & 0 \end{vmatrix} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 0 & 0 \end{vmatrix} = 0$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial y} & 0 \\ \frac{\partial Q}{\partial y} & -1 \end{vmatrix} = -\frac{\partial f}{\partial y}$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & -1 \end{vmatrix} = 0$$

Substitude these values in eq^n(2)

(2) $\Rightarrow \frac{\partial Q}{\partial x} + P(0) - \frac{\partial f}{\partial y} + q(0) = 0$

$\Rightarrow \frac{\partial Q}{\partial x} = \frac{\partial f}{\partial y}$

$\therefore$ The eq^n's $p=P(x,y)$ & $q=Q(x,y)$ are compatible if $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

Problems:-

1, Show that $p^2+q^2=1$ & $(p^2+q^2)x = pz$ are compatible and solve th

Sol:- Given $f(x,y,z,p,q) = p^2+q^2-1=0$ — (1)

$g(x,y,z,p,q) = p^2x+q^2x-pz=0$ — (2)

To prove (1) and (2) are compatible we have to prove that

$[f,g]=0$

ie, $\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$ — (3)

Now $\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & 2p \\ p^2+q^2 & 2px-z \end{vmatrix}$

$= -2p(p^2+q^2)$

$= -(2p^3+2pq^2)$

$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} 0 & 2p \\ -p & 2px-z \end{vmatrix}$

$= -2p(-p) = 2p^2$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 2q \\ 0 & 2qx \end{vmatrix} = 0$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & 2q \\ -p & 2qx \end{vmatrix}$$

$= -2q(-p) = 2pq$

Substitude all these values in eq^n (3)

$\Rightarrow -2p^3 - 2pq^2 + p(2p^2) + 0 + q(2pq)$

$\Rightarrow 0$

$\therefore$ Given eq^n's are compatible.

Now solve (1) & (2) $p^2+q^2=1$ and $(p^2+q^2)x = pz$

$\Rightarrow 1x = pz$

$\Rightarrow p = \frac{x}{z}$

Substitude 'p' in $p^2+q^2=1$

$\Rightarrow \frac{x^2}{z^2} + q^2 = 1$

$\Rightarrow q^2 = 1 - \frac{x^2}{z^2}$

$\Rightarrow q^2 = \frac{(x^2-z^2)}{z^2}$

$\Rightarrow q = \frac{1}{z}\sqrt{(x^2-z^2)}$

Substitude p, q values in $dz = pdx + qdy$

$\Rightarrow dz = \frac{x}{z}dx + \frac{1}{z}\sqrt{(x^2-z^2)}dy$

$\Rightarrow xdx + \sqrt{(x^2-z^2)}\,dy = zdz$

$\Rightarrow \sqrt{z^2-x^2}\,dy = zdz - xdx$

$\Rightarrow dy = \frac{zdz - xdx}{\sqrt{z^2-x^2}}$

Put $z^2 - x^2 = t$

$2zdz - 2xdx = dt$

$zdz - xdx = \frac{dt}{2}$

Now $dy = \frac{dt}{2\sqrt{t}}$

$= \frac{1}{2}t^{-1/2}dt$

Integrate on both sides

$\frac{1}{2}\frac{t^{-1/2+1}}{-1/2+1} = y+c$

$\frac{\frac{1}{2}t^{1/2}}{\frac{1}{2}} = y+c$

$\sqrt{t} = y+c$

$\Rightarrow \sqrt{z^2-x^2} = y+c$

Squaring on both sides

$z^2 - x^2 = (y+c)^2$

$\therefore z^2 = (y+c)^2 + x^2$

Hence the sol^n.

2. Show that the eq^n $z = px + qy$ is compatible with any eq^n $f(x, y, z, p, q) = 0$ ie, homogenous in x, y & z. Solve completely homogenous eq^n's $z = px + qy$, $2xy(p^2+q^2) = z(yp + xy)$

Sol^n: Given the differential eq^n $f(x, y, z, p, q) = 0$ —(1) is homogenous function in x, y, z

Then clearly $f(x, y, z, p, q)$ will be a homogenous function in variables x, y, z say of degree 'n'

Then by Euler's theorem on homogenous functions, we have

$$x\left(\frac{\partial f}{\partial x}\right) + y\left(\frac{\partial f}{\partial y}\right) + z\left(\frac{\partial f}{\partial z}\right) = nf$$

Let us take $\frac{\partial f}{\partial y} = 0$ & by (1) we have $f = 0$

$$\Rightarrow x\left(\frac{\partial f}{\partial x}\right) + y \cdot 0 + z\left(\frac{\partial f}{\partial z}\right) = 0$$

$$\Rightarrow x\left(\frac{\partial f}{\partial x}\right) + z\left(\frac{\partial f}{\partial z}\right) = 0 \quad \text{—(2)}$$

Let $g(x, y, z, p, q) = px + qy - z = 0$ —(3)

Now we have to prove that (2) and (3) are compatible

ie, we have to show that $[f, g] = 0$

we know that $[f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$ —(4)

consider $\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial p} \\ p & x \end{vmatrix}$

$$= x\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial p}$$

Now $\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial p} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial p} \\ -1 & x \end{vmatrix}$

$$\therefore \frac{\partial(f,g)}{\partial(z,p)} = x\frac{\partial f}{\partial z} + \frac{\partial f}{\partial p}$$

now, $\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ q & y \end{vmatrix}$

$$\therefore \frac{\partial(f,g)}{\partial(y,q)} = y\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial q}$$

now, $\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ -1 & y \end{vmatrix}$

$$\therefore \frac{\partial(f,g)}{\partial(z,q)} = y\frac{\partial f}{\partial z} + \frac{\partial f}{\partial q}$$

Now, Substitude all these values in (4) we get

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)}$$

$$= x\frac{\partial f}{\partial x} - \cancel{p\frac{\partial f}{\partial p}} + px\frac{\partial f}{\partial z} + \cancel{p\frac{\partial f}{\partial p}} + y\frac{\partial f}{\partial y} - \cancel{q\frac{\partial f}{\partial q}} + qy\frac{\partial f}{\partial z} + \cancel{q\frac{\partial f}{\partial q}}$$

$$= x\frac{\partial f}{\partial x} + px\frac{\partial f}{\partial z} + y\frac{\partial f}{\partial y} + qy\frac{\partial f}{\partial z}$$

$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + (px+qy)\frac{\partial f}{\partial z}$$

we know that $px+qy=z$

$$= x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z}$$

$= n \cdot f$

$= n \cdot 0$

$\therefore [f,g] = 0$

Hence the differential eq^n $z = px+qy$ is compatible with any diff-eq^n $f(x,y,z,p,q)=0$ ie, homogenous in x, y and z.

Q. Let $z = px+qy$ — (1)

and $2xy(p^2+q^2) - z(yp+xy) = 0$ — (2)

Let $f(x,y,z,p,q) = 2xy(p^2+q^2) - z(yp+xq) = 0$

$\therefore 2xy(p^2+q^2) - z(yp+xq) = 0$ is homogenous.

we know that $z = px+qy$ is compatible with homogenous eq^n

$\therefore$ (1) and (2) are compatible.

From eq^n (1) we have, $px+qy=z$

$$\Rightarrow q = \frac{z-px}{y} \quad \text{— (3)}$$

Substitude this value in (2), we get

$$2xy\left[p^2 + \left(\frac{z-px}{y}\right)^2\right] - z\left[yp + x\left(\frac{z-px}{y}\right)\right] = 0$$

$$\Rightarrow 2x\cancel{y}\left(\frac{p^2y^2+(z-px)^2}{y\cancel{y}}\right) - z\left[\frac{y^2p + x(z-px)}{y}\right] = 0$$

$$\Rightarrow 2x(p^2y^2 + (z-px)^2) - z(y^2p + xz - px^2) = 0$$

$$\Rightarrow 2x(p^2y^2 + z^2 + p^2x^2 - 2pzx) - z(y^2p + xz - px^2) = 0$$

$$\Rightarrow 2xp(x^2p + y^2p) - z(3x^2p + y^2p) + xz^2 = 0$$

$$\Rightarrow 2xp(x^2p + y^2p) - 3zx^2p - y^2zp + xz^2 = 0$$

$\Rightarrow 2xp(x^2p+y^2p) - 2zx^2p - x^2zp - y^2zp + xz^2 = 0$

$\Rightarrow 2xp(x^2p+y^2p) - zp(x^2+y^2) - xz(2xp-z) = 0$

$\Rightarrow (2xp - zp)(x^2p+y^2p) - xz(2xp-z) = 0$

$\Rightarrow (2xp-z)((x^2p+y^2p) - xz) = 0$

$\Rightarrow 2xp - z = 0 \; ; (x^2+y^2)p - xz = 0$

$\Rightarrow (x^2+y^2)p = xz \quad ; \quad 2xp = z$

$\Rightarrow p = \dfrac{xz}{x^2+y^2} \quad ; \quad p = \dfrac{z}{2x}$

Substitude $p = \frac{z}{2x}$ in $q = \frac{z - px}{y}$

$\Rightarrow q = \dfrac{z - \left(\frac{z\cancel{x}}{2\cancel{x}}\right)}{y}$

$\Rightarrow q = \dfrac{z}{2y}$

Substitude $p = \frac{xz}{x^2+y^2}$ in $q = \frac{z-px}{y}$

Now $q = \dfrac{z - \left(\frac{xz}{x^2+y^2}\right)x}{y}$

$q = \dfrac{z\cancel{x^2} + zy^2 - \cancel{x^2z}}{y(x^2+y^2)}$

$q = \dfrac{zy^2}{y(x^2+y^2)} \Rightarrow q = \dfrac{zy}{x^2+y^2}$

Substitude $p = \frac{z}{2x}$; $q = \frac{z}{2y}$ in $dz = pdx + qdy$

$\Rightarrow dz = \dfrac{z}{2x}dx + \dfrac{z}{2y}dy$

$\Rightarrow dz = \dfrac{z}{2}\left(\dfrac{1}{x}dx + \dfrac{1}{y}dy\right)$

$\Rightarrow \dfrac{2dz}{z} = \dfrac{1}{x}dx + \dfrac{1}{y}dy$

Integrate on both sides

$\Rightarrow 2\log z = \log x + \log y + \log c_1$

$\Rightarrow z^2 = xyc_1 \quad \text{---} (4)$

Substitude $p = \frac{xz}{x^2+y^2}$ & $q = \frac{zy}{x^2+y^2}$ in $dz = pdx + qdy$

$dz = \dfrac{xz}{x^2+y^2}dx + \dfrac{zy}{x^2+y^2}dy$

$dz = z\left[\dfrac{xdx}{x^2+y^2} + \dfrac{ydy}{x^2+y^2}\right]$

$$\frac{dz}{z} = \frac{x\,dx + y\,dy}{x^2+y^2}$$

$$\frac{dz}{z} = \frac{1}{2}\frac{(2x\,dx + 2y\,dy)}{x^2+y^2}$$

$$2\frac{dz}{z} = \frac{2x\,dx + 2y\,dy}{x^2+y^2}$$

Integrate on both sides

$$2\log z = \log(x^2+y^2) + \log c_2$$

$$z^2 = (x^2+y^2)\,c_2 \quad \text{---(5)}$$

$\therefore$ (4) and (5) are req. solns of given eqn.

Introduction:- SEC-10 CHARPIT'S METHOD

Let the given P.D.E. of first order and non-linear in p, q be

$f(x,y,z,p,q)=0$ —(1)

The fundamental ideal in charpit's method is the introduction of a second order P.D.E. of first order $g(x,y,z,p,q)=0$ —(2) which contains the arbitrary constant 'a'.

i, By solving (1) and (2) we get $p=p(x,y,z,a)$ and $q=q(x,y,z,a)$

ii, Now substitude p and q values in $dz=pdx+qdy$ —

then $dz=p(x,y,z,a)\,dx+q\,(x,y,z,a)\,dy$ —(3) is integrable when such a function 'g' has been found in the solⁿ of eqⁿ (3) is $F(x,y,z,a,b)=0$ —(4) is the solⁿ of eqⁿ (1) with two arbitiary constants 'a' and 'b'

Procedure:-

i, Transfer all terms of the given eqⁿ to L.H.S. and denote the entire expression by f

ii, write down the charpit's auxilary eqⁿ

$$\frac{-dp}{f_x+pf_z}=\frac{-dq}{f_y+qf_z}=\frac{dz}{pf_p+qf_q}=\frac{dx}{f_p}=\frac{dy}{f_q}$$

where $f_x=\frac{\partial f}{\partial x}$; $f_y=\frac{\partial f}{\partial y}$; $f_z=\frac{\partial f}{\partial z}$; $f_p=\frac{\partial f}{\partial p}$; $f_q=\frac{\partial f}{\partial q}$

iii, After simplyfing select to proper functions so that the resolution integral may come out to be simplest relation involving atleast one of p and q.

iv, Put the values of p and q in $dz=pdx+qdy$

By integrating we get the complete integral of the given eqⁿ

Problems:-

① Find the complete integral of $p^2x+q^2y=z$

Sol: Given eq^n :- $p^2x+q^2y=z \Rightarrow p^2x+q^2y-z=0$ —— (I)

Let $f=p^2x+q^2y-z$ —— (1)

Now we have to find the req. values for the charpit's auxiliary eq^n.

Now, $f_x=p^2$

$f_y=q^2$, $f_z=-1$, $f_p=2px$, $f_q=2qy$

we know that the charpit's auxilary eq^n is

$$\frac{-dp}{f_x+pf_z}=\frac{-dq}{f_y+qf_z}=\frac{dz}{pf_p+qf_q}=\frac{dx}{f_p}=\frac{dy}{f_q} \quad \text{—— (2)}$$

substitude all the above values in (2)

$$\frac{-dp}{p^2+p(-1)}=\frac{-dq}{q^2+q(-1)}=\frac{dz}{p(2px)+q(2qy)}=\frac{dx}{2px}=\frac{dy}{2qy}$$

$$\frac{-dp}{p^2-p}=\frac{-dq}{q^2-q}=\frac{dz}{2p^2x+2q^2y}=\frac{dx}{2px}=\frac{dy}{2qy} \quad \text{—— (3)}$$

From eq^n(3) consider, $\frac{dp}{p^2-p}$ & $\frac{dx}{2px}$ terms as L.H.S. and multiply $2px$ and p^2 and add them

Again from (3) consider, $\frac{dq}{q^2-q}$ & $\frac{dy}{2qy}$ terms as R.H.S, multiply $2qy$ and q^2 and add them.

$$\left(\because \frac{dp}{p^2-p}, \frac{dx}{2px}\right.$$

$$\frac{dp}{p^2-p}+\frac{dx}{2px}=0$$

$$\frac{dp}{p^2-p}-\frac{dx}{2px}=0$$

$$\frac{dp}{p^2-p}+\frac{dx}{-2px}=0$$

$$\frac{2xp\,dp+p^2dx}{2px(p^2-p)+p^2(-2px)}=\frac{2qy\,dq+q^2dy}{2qy(q^2-q)+q^2(2qy)}$$

$$\frac{d(p^2x)}{\cancel{2p^3x}-2xp^2\cancel{(+2p^3x)}}=\frac{d(q^2y)}{\cancel{2q^3y}-2q^2y\cancel{(2q^3y)}}$$

$$\frac{d(p^2x)}{\cancel{-}2xp^2}=\frac{d(q^2y)}{\cancel{-}2q^2y}$$

$$\frac{d(p^2x)}{p^2x}=\frac{d(q^2y)}{q^2y}$$

Integrate on both sides $\Rightarrow \log p^2x=\log q^2y+\log a$

$p^2x=q^2ya$ —— (4)

From (I) we have, $p^2x+q^2y-z=0$

$q^2ya+q^2y-z=0$

$q^2y(1+a)-z=0$

$$q^2 y = \frac{z}{(1+a)} \Rightarrow q^2 = \frac{z}{(1+a)y} \quad -(5) \Rightarrow q = \left[\frac{z}{(1+a)y}\right]^{1/2}$$

Now substitude q^2 value in (4)

$$p^2 x = \frac{z}{(1+a)\cancel{y}}(\cancel{y}a)$$

$$p^2 x = \frac{za}{1+a}$$

$$p^2 = \frac{za}{(1+a)x}$$

$$p = \left[\frac{za}{(1+a)x}\right]^{1/2} \quad -(6)$$

Substitude p and q in $dz = pdx + qdy$

now $dz = \left[\frac{za}{(1+a)x}\right]^{1/2} dx + \left[\frac{z}{(1+a)y}\right]^{1/2} dy$

$$dz = \frac{a^{1/2} z^{1/2}}{x^{1/2}(1+a)^{1/2}} dx + \frac{z^{1/2}}{(1+a)^{1/2} y^{1/2}} dy$$

$$dz = \frac{z^{1/2}}{(1+a)^{1/2}}\left[\frac{a^{1/2}}{x^{1/2}} dx + \frac{1}{y^{1/2}} dy\right]$$

$$\frac{(1+a)^{1/2}}{z^{1/2}} dz = a^{1/2} x^{-1/2} dx + y^{-1/2} dy$$

$$(1+a)^{1/2} z^{-1/2} dz = a^{1/2} x^{-1/2} dx + y^{-1/2} dy$$

$\left(\int x^{-1/2} dx = \frac{x^{-1/2+1}}{-1/2+1} = \frac{x^{1/2}}{1/2} = 2x^{1/2}\right)$

Now integrate on both sides

$$(a+1)^{1/2}\, 2z^{1/2} = a^{1/2}\, 2x^{1/2} + 2y^{1/2} + 2b$$

$$\cancel{2}\left[(a+1)^{1/2} z^{1/2}\right] = \cancel{2}(a^{1/2} x^{1/2} + y^{1/2} + b]$$

$\therefore (z(a+1))^{1/2} = (ax)^{1/2} + (y^{1/2}+b)$ is the req. solⁿ of given eqⁿ.

2. Find the complete integral of $f = 2z + p^2 + qy + 2y^2 = 0$ by charpit's method.

sol:- given eqⁿ :- $f = 2z + p^2 + qy + 2y^2 = 0$

Let $f = 2z + p^2 + qy + 2y^2 \quad -(1)$

Now we have to find the req. values for the charpit's auxiliary eqⁿ

Now, $f_x = 0$; $f_y = q + 4y$; $f_z = 2$; $f_p = 2p$; $f_q = y$

we know that charpit's auxilary eqⁿ is

$$\frac{-dp}{f_x + pf_z} = \frac{-dq}{f_y + qf_z} = \frac{dz}{pf_p + qf_q} = \frac{dx}{-f_p} = \frac{dy}{f_q}$$

$$\frac{-dp}{0+p(2)} = \frac{-dq}{q+4y+q(2)} = \frac{dz}{P(2p)+q(y)} = \frac{dx}{2p} = \frac{dy}{y}$$

$$\frac{-dp}{2p} = \frac{-dq}{3q+4y} = \frac{dz}{2p^2+qy} = \frac{dx}{2p} = \frac{dy}{y} \quad —(2)$$

From (2) consider first and fourth terms

$$\frac{-dp}{\cancel{2}p} = \frac{dx}{\cancel{2p}}$$

$$dp + dx = 0$$

Integrate on both sides

$$p + x = a$$

$$p = a - x$$

Substitude 'p' in eq^n (1)

$$2z + (a-x)^2 + qy + 2y^2 = 0$$

$$\Rightarrow qy = -2y^2 - 2z - (a-x)^2$$

$$\Rightarrow q = -\frac{2y^{\cancel{2}}}{\cancel{y}} - \frac{2z}{y} - \frac{(a-x)^2}{y}$$

$$\Rightarrow q = -2y - \frac{2z}{y} - \frac{(a-x)^2}{y} \quad —(3)$$

Substitude p, q in $dz = pdx + qdy$

$$dz = (a-x)dx + \left(-2y - \frac{2z}{y} - \frac{(a-x)^2}{y}\right)dy$$

$$y\,dz = (a-x)y\,dx + (-2y^2 - 2z - (a-x)^2)dy$$

$$y\,dz + 2z\,dy = (a-x)y\,dx - 2y^2dy - (a-x)^2dy$$

Multiply with 'y' on both sides

$$y^2dz + 2yz\,dy = (a-x)y^2dx - 2y^3dy - (a-x)^2y\,dy$$

$$d(y^2z) = \frac{1}{2}\left[2y^2(a-x)dx - 2y(a-x)^2dy\right] - 2y^3dy$$

$$\left[\because d(y^2(a-x)^2) = 2y(a-x)^2 \cancel{dy}\,dy + y^2 2(a-x)(-1)dx\right.$$

$$\left. = 2y(a-x)^2dy - 2y^2(a-x)dx\right]$$

$$d(y^2z) = \frac{1}{2}\left[-d(y^2(a-x)^2)\right] - 2y^3dy$$

Integrate on both sides

$$y^2z = -\frac{1}{2}y^2(a-x)^2 - \frac{2y^4}{4} + b$$

$$z = -\frac{\cancel{y^2}(a-x)^2}{2\cancel{y^2}} - \frac{y^4}{2y^2} + \frac{b}{y^2} \quad \text{where 'b' is constant}$$

$$z = -\frac{(a-x)^2}{2} - \frac{1}{2}y^2 + \frac{b}{y^2}$$

$$z = \frac{1}{2}\left[-(a-x)^2 + y)^2\right] + b$$

$2z = -(a-x)^2 - y^2 + b$ is the req. sol^n of given eq^n.

Sec-11 Special Types Of First Order Equations

In this section we shall consider some special types of first Order partial differential eqn's whose solutions may be obtained early by charpit's method.

Method:- 1 Equations involving 'p' and 'q' :-

For equations of the type $f(p,q)=0$ — (1)

The charpit's eqn reduces to $\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-qf_q}=\frac{dp}{0}=\frac{dq}{0}$

$$\Rightarrow \frac{dx}{f_p}=\frac{dy}{f_q}=\frac{dz}{pf_p+qf_q}=\frac{dp}{0}=\frac{dq}{0}$$

Now $\frac{dp}{0}=\frac{dq}{0}$

Taking the first term $dp=0$

Then by integrating we have $p=a$ — (2) where a is constant

Substitude the value of 'p' in eqn (1)

$f(a,q)=0$ — (3)

Take $dq=0$

$\Rightarrow q=b$ where 'b' is constant

Substitude 'b' in (3) $f(a,b)=0$ — (4)

Now substitude 'p' and 'q' values in $dz=pdx+qdy$

$$dz=adx+\cancel{q}^{\,b}dy$$

Now integrating on both sides we get $z=ax+by+c$ — (5)

where 'c' is arbitrary constant

Now solving (4) we get $b=F(a)$

Substitude these values in eqn (5) we get

$$z=ax+F(a)y+c \quad \text{— (6)}$$

which contains two arbitrary constants 'a' and 'c' which are equal to the number of independent variables 'x' and 'y'

The singular integrals of eqn (1) is obtained by eliminating 'a' & 'c' between the complete integral of eqn (6) and the eqn obtained by differentiating eqn (6) partially with respect to 'a' & 'c'

we get $z=ax+by\ F(a)y+c$

Now differentiate with respect to 'a'

$\Rightarrow 0=x+yF'(a)$

Differentiate with respect to 'c'

$\Rightarrow 0 = 1$

If $z = ax + F(a)y + c$ then we have $\left.\begin{array}{l} 0 = x + yF'(a) \\ 0 = 1 \end{array}\right\} - (7)$

Since the last eqn in eq^n (7) is meaning less (ie, $0 = 1$). we can conc-lude that the eq^n of standard form of (1)

[∵ only p and q are present have no singular solution] To find the general sol^n of (1), first we take $c = \phi(a)$ in eq^n (6)

we get $z = ax + F(a)y + \phi(a)$ — (8)

where ϕ is arbitrary constant

Now differentiate eq^n (8) partially with respect to 'a' we get

$$0 = x + yF'(a) + \phi'(a) \text{ —— } (9)$$

By eliminating the 'a' b/w (8) and (9) we get the gener-al sol^n of eq^n(1)

Problems:-

1. Find the complete integral of the eq^n $pq = 1$

Given eq^n is $pq = 1$ — (1)

It is in the form of eq^n's involving only p and q

Here $p = a$

Sub. $p = a$ in (1) ie, $qa = 1$

$$\Rightarrow q = 1/a \quad (\because q = F(a))$$

∴ The complete integral is $z = ax + yF(a) + c$

$$z = ax + y/a + c$$

$$\Rightarrow az = a^2x + y + ac$$

Let $ca = c_1$

$$\Rightarrow az = a^2x + y + c_1$$

where 'a' and 'c' are arbitrary constants.

2. Find the complete integral of $p + q = pq$

Given eq^n is $p + q = pq$ — (1)

It is in the form of the eq^n involving p and q

Here $p = a$

Sub. $p = a$ in (1) then (1) becomes

$$a + q = aq$$

$$\Rightarrow a = aq - q$$

$$\Rightarrow a = q(a - 1)$$

$$\Rightarrow q = \frac{a}{a-1}$$

The complete integral is $z = ax + yF(a) + c$

$$\Rightarrow z = ax + y\frac{a}{a-1} + c$$

$$z(a-1) = a(a-1)x + ya + c(a-1)$$

$$a(a-1)x + ay - z(a-1) = c_1$$

where a and c are arbitrary constants.

Method:-2 Equations not involving the independent Variables

In this method the partial diff - al eqⁿ is of the type

$f(z,p,q) = 0$ — (1) $F(x,y,z,p,q)=0 \rightarrow$ This is 1st order PDE
Here x and y are independent variables so $f(z,p,q)$

we know that the charpit's auxilary eqⁿ is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p + qf_q} = \frac{-dp}{pf_z} = \frac{-dq}{qf_z}$$

consider last two terms

$$\Rightarrow \frac{\cancel{-}dp}{p\cancel{f_z}} = \frac{\cancel{-}dq}{q\cancel{f_z}}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q}$$

Integrating on both sides, we have $\log p = \log q + \log a$

$$p = qa \quad — (2)$$

Solving (1) and (2) we obtain expressions for p, q which is a complete integral of (1)

1, Find a complete integral of the eqⁿ $p^2z^2 + q^2 = 1$

Given eqⁿ is $p^2z^2 + q^2 = 1$ — (1)

Substitude $p = aq$

$$a^2q^2z^2 + q^2 = 1$$

$$\Rightarrow q^2(a^2z^2 + 1) = 1$$

$$\Rightarrow q^2 = \frac{1}{1 + a^2z^2}$$

$$\Rightarrow q = \frac{1}{\sqrt{1 + a^2z^2}}$$

$$\Rightarrow q = (1 + a^2z^2)^{-1/2}$$

Substitude q in $p = aq$

$$\Rightarrow p = a(1 + a^2z^2)^{-1/2}$$

Substitude 'p' and 'q' in $dz = pdx + qdy$

$$dz = a(1+a^2z^2)^{-1/2}dx + (1+a^2z^2)^{-1/2}dy$$

$$dz = (1+a^2z^2)^{-1/2}(adx + dy)$$

$$(1+a^2z^2)^{1/2}dz = adx + dy$$

Integrating on both sides we get

$$\frac{2}{3}(1+a^2z^2)^{3/2} = ax+y+b$$

$$\therefore (1+a^2z^2)^{3/2} = \frac{3}{2}(ax+y+b)$$

Method:-3 Separable Equations

we say that a first order P.D.E is separable if it can be written in the form $f(x,p) = g(y,q)$ for such a eqn charpit's eqn becomes as

$$\frac{dx}{f_p} = \frac{dy}{-g_q} = \frac{dz}{pf_p - qg_q} = \frac{dp}{-f_x} = \frac{dq}{-g_y}$$

consider $\frac{dx}{f_p} = \frac{dp}{-f_x}$

So that we have an ordinary differential eqn $\frac{dp}{dx} + \frac{f_x}{f_p} = 0$

in x and p which may be solved to give p as a function of 'x' and an arbitrary constant 'a' writing this eqn in the form

$f_p\,dp + f_x\,dx = 0$

we see that its soln is $f(x,p) = a$

Hence we determine p, q from the relations $f(x,p) = a$, $g(y,q) = a$ then proceed as in the general theory.

Problems:-

1, Find a complete integral of the eqn $p^2y(1+x^2) = qx^2$

sol:- Given eqn is $p^2y(1+x^2) = qx^2$ — (1)

It can be written in the form $f(x,p) = g(y,q)$

$$\frac{p^2(1+x^2)}{x^2} = \frac{q}{y}$$

Here $f(x,p) = \frac{p^2(1+x^2)}{x^2}$ and $g(y,q) = \frac{q}{y}$

Now, $f_x = p^2\frac{d}{dx}\left(\frac{1+x^2}{x^2}\right)$

$$= \frac{p^2(x^2(2x) - (1+x^2)2x)}{x^4}$$

$$= \frac{p^2(\cancel{2x^3} - 2x - \cancel{2x^3})}{x^4}$$

$$= \frac{p^2(-2x)}{x^4} = \frac{-2p^2}{x^3}$$

$f_p = \frac{1+x^2}{x^2}\frac{d}{dp}(p^2)$

$$= \frac{2p(1+x^2)}{x^2}$$

$f_p\,dp + f_x\,dx = 0$

$\frac{2p(1+x^2)}{x^2}\,dp + \left(\frac{-2p^2}{x^3}\right)dx = 0$

$\frac{(1+x^2)}{x^2}\,2p\,dp - 2p^2\left(\frac{1}{x^3}\right)dx = 0$

$\frac{(1+x^2)}{x^2}\cancel{2}\left(\frac{p^2}{\cancel{2}}\right) - \cancel{2}p^2\left(\frac{x^{-2}}{-\cancel{2}}\right) = a^2$

$\frac{(1+x^2)p^2}{x^2} + \frac{p^2}{x^2} = a^2$

$(1+x^2)p^2 + p^2 = a^2x^2$

Method-4 Clairaut Equation:-

A first order partial differential eqn is said to be clairaut type if it can be written in the form $z = px + qy + f(p,q)$ —(1) then the corresponding charpit's eqn is

$$\frac{dx}{f_p + x} = \frac{dy}{y + f_q} = \frac{dz}{px + qy + pf_p + qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

Since $dp = 0$ & $dq = 0$ and integrate we get $p = a$ and $q = b$ substitute these values in ① to get the complete integral ie.,

$z = ax + by + f(a,b)$

1, $(p+q)(z - xp - yq) = 1$

Given eqn is $(p+q)(z - xp - yq) = 1$

$\Rightarrow z - xp - yq = \frac{1}{p+q}$

$\Rightarrow z = \frac{1}{p+q} + xp + yq$ —(2)

It is in the form $z = px + qy + f(p,q)$

Substitude $p = a$ & $q = b$ in (2)

$z = \frac{1}{a+b} + ax + by$ is req. complete integral.

, $pqz = p^2(xq + p^2) + q^2(yp + q^2)$

Given eq^n is $pqz = p^2(xq + p^2) + q^2(yp + q^2)$ —(1)

$pqz = p^2xq + p^4 + q^2yp + q^3 \cdot q$

$$z = \frac{pq(px+qy)}{pq} + \frac{p^4+q^4}{pq}$$

$$z = px + qy + \frac{p^4+q^4}{pq} \quad \text{—(2)}$$

It is in the form $z = px + qy + f(p,q)$

substitude $p = a$ & $q = b$ in (2)

$z = ax + by + f(a,b)$ is req. complete integral.

SEC-12 SOLUTIONS SATISFYING GIVEN CONDITIONS

In this section we shall consider the determination of surfaces whi-ch satisfies the P.D.E $f(x,y,z,p,q) = 0$ —(1) and which satisfy so-me other condition such as passing through a given curve & circumscribing a we shall also consider how to determine the sol^n of (1) which passes through a given curve 'C' which has a parametric eq^n $x = x(t)$, $y = y(t)$, $z = z(t)$ —(2) being a parameter if there is an integral surface of the eq^n (1) prove the curve 'C' then it is

a, A particular case of the complete integral $f(x,y,z,a,b) = 0$ —(3) obtained by giving particular values to 'a' & 'b'

(or)

b, A particular case of the general integral corresponding to eq^n(3)

c, the envelope of a one parameter sub system of (3)

(or)

the envelope of the two parameter system eq^n (3)

It seems unlikely that the sol^n fall into either case 'a' or case 'c' so we considered the case b, which is the one which occurs most frequently.

$\therefore$ we suppose that a surface 'E' say of type B exists and passes through the curve 'C'. At every one of its points this envelope is touched by same number of the subsystem in particular at each point 'p' of the curve 'c' we may suppose 'E' to be to-ed by a number S_p (say) of the sub-system and since S_p touches E at p. It also touches 'c' at the same point.

In other words E is the envelope of a one parameter Sub system of ③ each of whose members touches the curve C, provided that such a sub-system exists to determine 'E', must consider the subsystem made up of those members of the family 3 which touch the curve 'c'. The points of intersection of the surface 3 and the curve 'c' are determine in terms of the parameter 3. By the eqn $f(x(t), y(t), z(t), a, b) = 0$ (4) and the condition that the curve 'c' should touch the surface ③ is that the eqn ④ must have two equal roots or what is the same thing, that eqn ④ and the eqn

$\frac{d}{dt} f(x(t), y(t), z(t), a, b) = 0$ —⑤ should have a common root

The condition for this to be so, is the element of 't' from ④ and ⑤ is $\varphi(a,b) = 0$ —(6)

which is a relation b/w 'a' and 'b' allow the eqn ⑥ to be factorized into a set of alternating equations

$b = \phi_1(a),\ b = \phi_2(a), \text{------}$ (7)

Each of which defines a subsystem of one parameter, envelope of each of these one parameter subsystem is a solution of the problem.

**(a) Find a complete integral of the P.D.E. $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x=0,\ z^2=4y$

Sol:- Given PDE is $(p^2+q^2)x = pz$ —(1)

Let $f = (p^2+q^2)x - pz = 0$

ie, $f = p^2x + q^2x - pz$

we know that charpit's auxilary eqn is

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{pf_p+qf_q} = \frac{-dp}{f_x+pf_z} = \frac{-dq}{f_y+qf_z}$$

$f_x = p^2+q^2,\ f_y = 0,\ f_z = -p,\ f_p = 2px - z,\ f_q = 2qx$

$$\frac{dx}{2px-z} = \frac{dy}{2qx} = \frac{dz}{-2p^2x - pz + 2q^2x} = \frac{-dp}{\cancel{p^2}+q^2-\cancel{p^2}} = \frac{\cancel{+}dq}{\cancel{+}pq}$$

consider, $\dfrac{dp}{q^{\cancel{2}}} = \dfrac{\cancel{+}dq}{\cancel{+}p\cancel{q}}$

$pdp = -qdq \Rightarrow pdp + qdq = 0$

$\dfrac{p^2}{2} + \dfrac{q^2}{2} = a^2/2$

$\Rightarrow p^2+q^2 = a^2$

substitude $p^2+q^2=a^2$ in (1)

$a^2x = pz$

$a^2x - pz = 0 \Rightarrow p = \dfrac{a^2x}{z}$

substitude 'p' value in $p^2+q^2=a^2$

$\dfrac{a^4x^2}{z^2} + q^2 = a^2$

$q^2 = a^2 - \dfrac{a^4x^2}{z^2}$

$q^2 = \dfrac{a^2z^2 - a^4x^2}{z^2}$

$q^2 = \dfrac{a^2(z^2-a^2x^2)}{z^2}$

$q = \dfrac{a}{z}\sqrt{z^2-a^2x^2}$

substitude 'p' & 'q' in $dz = pdx + qdy$

$dz = \dfrac{a^2x}{z}dx + \dfrac{a}{z}\sqrt{z^2-a^2x^2}\,dy$

$zdz = a^2x\,dx + a\sqrt{z^2-a^2x^2}\,dy$

$zdz - a^2x\,dx = a\sqrt{z^2-a^2x^2}\,dy$

$ady = \dfrac{zdz - a^2xdx}{\sqrt{z^2-a^2x^2}}$ —(3)

Let $z^2 - a^2x^2 = t$

$2zdz - 2a^2x\,dx = dt$

$zdz - a^2x\,dx = \dfrac{dt}{2}$

substitude this value in (3)

(3) $\Rightarrow ady = \dfrac{\frac{dt}{2}}{\sqrt{t}}$

$\dfrac{dt}{2\sqrt{t}} = ady$

$\dfrac{1}{2}\dfrac{dt}{\sqrt{t}} = ady$

Integrate on both sides

$\frac{1}{\cancel{2}}\cancel{2}\sqrt{t} = ay + b$

$\sqrt{t} = ay + b$

Squaring on both sides

$t = (ay+b)^2$

$z^2 - a^2x^2 = (ay+b)^2$

$z^2 = a^2x^2 + (ay+b)^2$ is the req. solⁿ.

SEC-13 JACOBI'S METHOD

Let $f(x,y,z,p,q)=0 \rightarrow (1)$ be a non-linear first order P.D.E. in two independent variables x, y. Let the solⁿ of eqⁿ (1) be of the form $u(x,y,z)=0$ —(2) and x,y,z be independent variables.

By treating 'u' as a dependent variable and x,y,z be independent variables, differentiating eqⁿ (2) partially with respect to x (with respect to 'x')

we get $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0$ (with respect to 'x')

with respect to 'y' $\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$

$\Rightarrow P_1 + P_3 p = 0$ & $P_2 + P_3 q = 0$ —(3)

where $P_1 = \frac{\partial u}{\partial x}$; $P_2 = \frac{\partial u}{\partial y}$ & $P_3 = \frac{\partial u}{\partial z}$

$p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$

Also $x = x_1$, $y = x_2$ & $z = x_3$ —(4)

From (3) we have $P_1 + P_3 p = 0$ & $P_2 + P_3 q = 0$

$$\Rightarrow p = \frac{-P_1}{P_3} \text{ \& } q = \frac{-P_2}{P_3} \quad \text{—(5)}$$

By using eqⁿ's (4) & (5), eqⁿ (2) reduces to

$f(x_1, x_2, x_3, P_1, P_2, P_3) = 0$

Now Jacobi's auxilary eqⁿ is

$$\frac{dP_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{\frac{-\partial f}{\partial P_1}} = \frac{dP_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{\frac{-\partial f}{\partial P_2}} = \frac{dP_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{\frac{-\partial f}{\partial P_3}} \quad \text{—(7)}$$

Now solving these eqⁿ's we obtain two eqⁿ's

$F_1(x_1, x_2, x_3, P_1, P_2, P_3) = a_1$ —(8)

$F_2(x_1, x_2, x_3, P_1, P_2, P_3) = a_2$ —(9) where a_1, a_2 are arbitrary constants.

Now we verify that the relations (8) and (9) satisfies the cond.

$$(F_1, F_2) = \Sigma\left(\frac{\partial f_1}{\partial x_r}\cdot\frac{\partial f_2}{\partial P_r} - \frac{\partial f_1}{\partial P_r}\frac{\partial f_2}{\partial x_r}\right) = 0 \quad \text{— (10)}$$

If eq^n (10) is satisfied then solve the eq^n's (8) & (9) then we get the values of P_1, P_2 & P_3

$\Rightarrow du = P_1 dx_1 + P_2 dx_2 + P_3 dx_3$

By integrating we get the complete sol^n of the given eq^n.

1, solve $p^2x + Q^2y = z$ by using Jacobi's method.

Given eq^n is $p^2x + Q^2 y = z$ —(1)

Let $U(x, y, z) = 0$ —(2) be the sol^n form of eq^n (1)

Treating 'U' as dependent variable and x, y, z as independent variables.

Now differentiate eq^n(2) partially with respect to x

ie, $\frac{\partial u}{\partial x} + \frac{\partial U}{\partial z}\frac{\partial z}{\partial x} = 0$

with respect to 'y' ie, $\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$

$\Rightarrow P_1 + P_3 p = 0$ & $P_2 + P_3 q = 0$ — (3)

where $P_1 = \frac{\partial U}{\partial x}$; $P_2 = \frac{\partial u}{\partial y}$; $P_3 = \frac{\partial U}{\partial z}$; $\frac{\partial z}{\partial x} = p$; $\frac{\partial z}{\partial y} = q$

Also $x = x_1$, $y = x_2$ & $z = x_3$ —(4)

From (3) we have $p = \frac{-P_1}{P_3}$ & $q = -\frac{P_2}{P_3}$ —(5)

Substitude (4) & (5) in (1) ie, $p^2x + Q^2 y = z$ ($\because Q = q$)

$$\Rightarrow \frac{P_1^2}{P_3^2}x + \frac{P_2^2}{P_3^2}y = z$$

$$\Rightarrow \frac{P_1^2}{P_3^2}x_1 + \frac{P_2^2}{P_3^2}x_2 = x_3$$

$$P_1^2x_1 + P_2^2 x_2 = P_3^2 x_3$$

$$\Rightarrow P_1^2x_1 + P_2^2x_2 - P_3^2x_3 = 0 \quad \text{— (6)}$$

eq^n (6) is in the form $f(x_1, x_2, x_3, P_1, P_2, P_3) = 0$

ie, $f = P_1^2x_1 + P_2^2x_2 - P_3^2x_3 = 0$ — (7)

Now consider the Jacobi's auxilary eq^n is

$$\frac{dP_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{\frac{-\partial f}{\partial P_1}} = \frac{dP_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{\frac{-\partial f}{\partial P_2}} = \frac{dP_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{\frac{-\partial f}{\partial P_3}}$$

Now we find all the required values

$\frac{\partial f}{\partial x_1} = P_1^2$, $\frac{\partial f}{\partial x_2} = P_2^2$, $\frac{\partial f}{\partial x_3} = P_3^2$

$\frac{\partial f}{\partial P_1} = 2P_1 x_1$, $\frac{\partial f}{\partial x_3} = -P_3^2$, $\frac{\partial f}{\partial x_2} = +P_2^2$ $\frac{\partial f}{\partial P_3} = -2x_3 P_3$

Substitude all these values in eq^n (2)

$$\frac{dP_1}{P_1^2} = \frac{dx_1}{-2x_1P_1} = \frac{dP_2}{P_2^2} = \frac{dx_2}{-2P_2x_2} = \frac{dP_3}{-P_3^2} = \frac{dx_3}{2P_3x_3}$$

Taking first two terms

$$\frac{dP_1}{P_1^{\cancel{2}}} = \frac{dx_1}{-2x_1\cancel{P_1}}$$

$$2\frac{dP_1}{P_1} + \frac{dx_1}{x_1} = 0$$

Integrate on both sides

$$2\log P_1 + \log x_1 = \log a_1$$

$$P_1^2 x_1 = a_1 \Rightarrow P_1 = \sqrt{\frac{a_1}{x_1}}$$

Taking next two functions

$$\frac{dP_2}{P_2^{\cancel{2}}} = \frac{dx_2}{-2\cancel{P_2}x_2}$$

$$-2\frac{dP_2}{P_2} = \frac{dx_2}{x_2}$$

$$2\frac{dP_2}{P_2} + \frac{dx_2}{x_2} = 0$$

Integrate on both sides

$$2\log P_2 + \log x_2 = \log a_2$$

$$P_2^2 x_2 = a_2 \Rightarrow P_2 = \sqrt{\frac{a_2}{x_2}}$$

Substitude P_1, P_2 values in eq^n 6

$$\frac{a_1}{\cancel{x_1}}\cancel{x_1} + \frac{a_2}{\cancel{x_2}}\cancel{x_2} - P_3^2 x_3 = 0$$

$$\cancel{+}x_3 P_3^2 = \cancel{+}(a_1 + a_2)$$

$$P_3^2 = \frac{a_1+a_2}{x_3} \Rightarrow P_3 = \sqrt{\frac{a_1+a_2}{x_3}}$$

Now substitude P_1, P_2 & P_3 values in $du = P_1dx_1 + P_2dx_2 + P_3dx_3$

$$du = \sqrt{\frac{a_1}{x_1}}dx_1 + \sqrt{\frac{a_2}{x_2}}dx_2 + \sqrt{\frac{a_1+a_2}{x_3}}dx_3$$

$$du = \sqrt{a_1}\frac{1}{\sqrt{x_1}}dx_1 + \sqrt{a_2}\frac{1}{\sqrt{x_2}}dx_2 + \sqrt{a_1+a_2}\frac{1}{\sqrt{x_3}}dx_3$$

Integrate on both sides

$$u = \sqrt{a_1}\,2\sqrt{x_1} + \sqrt{a_2}\,2\sqrt{x_2} + \sqrt{a_1+a_2}\,2\sqrt{x_3} + c$$

$$u = 2(\sqrt{a_1x_1} + \sqrt{a_2x_2} + \sqrt{(a_1+a_2)x_3}\,] + c$$

Let us take $u=0$, $a_2 = 1$ & $x_1 = x$, $y = x_2$, $x_3 = z$

$$0 = 2(\sqrt{a_1x} + \sqrt{y} + \sqrt{(a_1+1)z}) + c$$

$\therefore \sqrt{a_1 x} + \sqrt{y} + \sqrt{(a_1+1)z} + c = 0$ is required complete integral.

solve $p^2+q^2=k^2$

given eq^n is $p^2+q^2=k^2$ —(1)

Let $u(x,y,z)=0$ —(2) be the sol^n form of eq^n(1)

Treating u as dependent Variable & x,y,z as independent Variables.

Now diff. eq^n (2) partially w.r to x & y

w.r to x: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$

w.r to y: $\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$

$\Rightarrow P_1 + P_3 p = 0$ & $P_2 + P_3 q = 0$ — (3)

where $P_1 = \frac{\partial u}{\partial x}$, $P_2 = \frac{\partial u}{\partial y}$ & $P_3 = \frac{\partial u}{\partial z}$, $\frac{\partial z}{\partial x} = p$ & $\frac{\partial z}{\partial y} = q$

Also $x = x_1$, $y = x_2$, $z = x_3$ —(4)

from (3) we have

$p = \frac{-P_1}{P_3}$ $q = \frac{-P_2}{P_3}$ — (5)

Sub (4) & (5) in (1) we have (1) $\Rightarrow p^2+q^2 = k^2$

$\Rightarrow \left(\frac{-P_1}{P_3}\right)^2 + \left(\frac{-P_2}{P_3}\right)^2 = k^2$

$\Rightarrow \frac{P_1^2+P_2^2}{P_3^2} = k^2$

$= P_1^2 + P_2^2 = P_3^2 k^2$

$= P_1^2 + P_2^2 - P_3^2 k^2 = 0$ — (6)

eq^n (6) is in the form of $(x_1, x_2, x_3, P_1, P_2, P_3) = 0$

$f(P_1^2 + P_2^2 - P_3^2 k^2 = 0$

Now Consider the JaCobi's Auxilary eq^n

$$\frac{dP_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{-\frac{\partial f}{\partial P_1}} = \frac{dP_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{-\frac{\partial f}{\partial P_2}} = \frac{dP_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{-\frac{\partial f}{\partial P_3}}$$

Now we find the required values

$$\frac{\partial f}{\partial x_1}=0 \quad \frac{\partial f}{\partial x_2}=0 \quad \frac{\partial f}{\partial x_3}=0, \quad \frac{\partial f}{\partial p_1}=2p_1, \quad \frac{\partial f}{\partial p_2}=2p_2 \quad \frac{\partial f}{\partial p_3}=-2p_3k^2$$

$$\frac{dp_1}{0}=\frac{dx_1}{-2p_1}, \quad \frac{dp_2}{0}=\frac{dx_2}{-2p_2}, \quad \frac{dp_3}{0}=\frac{dx_3}{-2p_3k^2}$$

Taking ~~sub~~ ① & ③ ~~forms~~ fractions

$$\frac{dp_1}{0}=\frac{\partial p_2}{0}$$

$dp_1=0 \quad dp_2=0$

I.O.B.S.

$p_1=a_1, \; p_2=a_2$

Sub these value in eq^n ⑥

$a_1^2+a_2^2-p_3^2k^2=0$

$p_3^2k^2=a_1^2+a_2^2$

$$p_3^2=\frac{a_1^2+a_2^2}{k^2}$$

$$p_3=\frac{1}{k}\sqrt{a_1^2+a_2^2}$$

Now. p_1, p_2, p_3 values in $du=p_1dx_1+p_2dx_2+p_3dx_3$

$$du=a_1dx_1+a_2dx_2+1/k\sqrt{a_1^2+a_2^2}\,dx_3$$

I.O.B.S.

$$u=a_1x_1+a_2x_2+1/k\sqrt{a_1^2+a_2^2}\cdot x_3+c$$

Let us take $u=0$ & $a_2=1$, $x=x_1$, $y=x_2$, $z=x_3$.

$$0=a_1x+1.y+1/k.\sqrt{(a_1^2+1)}.z+c$$

$\Rightarrow a_1x+y+z/k\cdot\sqrt{(a_1^2+1)}+c=0$ is the required solution

4.a Show that a complete integral of the eq^n $f\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)=0$ is $u=ax+by+\theta(a,b)z+c$ where a,b,c are arbitrary constants and $f(a,b,\theta)=0$

b, Find a C.I. of the eq^n $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y}\cdot\frac{\partial u}{\partial z}$

Sol: a, Given eq^n is $f\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}\right)=0$ — (1)

Let $\frac{\partial u}{\partial x}=p_1$, $\frac{\partial u}{\partial y}=p_2$ and $\frac{\partial u}{\partial z}=p_3$

Substitude these values in (1) $f(P_1, P_2, P_3)=0$ —(2)

we know that Jacobison's auxilary eq^n is

$$\frac{dP_1}{\frac{\partial f}{\partial x_1}} = \frac{dx_1}{\frac{-\partial f}{\partial P_1}} = \frac{dP_2}{\frac{\partial f}{\partial x_2}} = \frac{dx_2}{\frac{-\partial f}{\partial P_2}} = \frac{dP_3}{\frac{\partial f}{\partial x_3}} = \frac{dx_3}{\frac{-\partial f}{\partial P_3}}$$

Now we find the required values

$$\frac{\partial f}{\partial x_1}=0 ; \frac{\partial f}{\partial P_1}=1 ; \frac{\partial f}{\partial x_2}=0 ; \frac{\partial f}{\partial P_2}=1 ; \frac{\partial f}{\partial x_3}=0 ; \frac{\partial f}{\partial P_3}=1$$

$$\Rightarrow \frac{dP_1}{0} = \frac{dx_1}{-1} = \frac{dP_2}{0} = \frac{dx_2}{-1} = \frac{dP_3}{0} = \frac{dx_3}{-1}$$

Taking first and third eq^n's $\frac{dP_1}{0} = \frac{dP_2}{0}$

$\Rightarrow dP_1 = 0$ & $dP_2 = 0$

Integrate on both sides

$P_1 = a$ & $P_2 = b$

Substitude P_1, P_2 in (2)

ie, $f(a, b, P_3) = 0$

So that P_3 = a function of a, b

$P_3 = \theta(a,b)$ (say)

Since we have $du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy + \left(\frac{\partial u}{\partial z}\right)dz$

$du = P_1 dx + P_2 dy + P_3 dz$

Substitude P_1, P_2, P_3 values in above eq^n

$du = a dx + b dy + \theta(a,b) dz$

Again integrate on both sides

$u = ax + by + \theta(a,b)z + c$

where 'c' is a arbitrary constant. a, b and θ are connecting by a relation $f(a, b, \theta(a,b)) = 0$ (by (1), (2) & (3))

Given eq^n is $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} - \frac{\partial u}{\partial x} \cdot \frac{\partial u}{\partial y} \cdot \frac{\partial u}{\partial z} = 0$ —(4)

Let $P_1 = \frac{\partial u}{\partial x}$, $P_2 = \frac{\partial u}{\partial y}$ & $P_3 = \frac{\partial u}{\partial z}$

Substitude these values in (4)

$P_1 + P_2 + P_3 - P_1 P_2 P_3 = 0$ —(5)

compare (5) with $f(P_1, P_2, P_3) = 0$

$f(P_1, P_2, P_3) = P_1 + P_2 + P_3 - P_1 P_2 P_3 = 0$ —(6)

By (a) we have $u = ax + by + \theta(a,b)z + c$ —(7)

Sub. $P_1 = a$, $P_2 = b$ & $P_3 = \theta(a,b)$ in 6

$\Rightarrow a + b + \theta(a,b) - a \cdot b\,\theta(a,b) = 0$

$\Rightarrow \theta(a,b) - ab(\theta(a,b)) = -(a+b)$

$\Rightarrow \theta(a,b)(1-ab) = -(a+b)$

$\Rightarrow \theta(a,b)(ab-1) = a+b$

$\therefore \theta(a,b) = \dfrac{a+b}{ab-1}$

substitude $\theta(a,b)$ in 7

ie, $u = ax + by + \left(\dfrac{a+b}{ab-1}\right)z + C$ is the required complete integral of 1